Page From a Physicist's Notebook

Yaw – Going round in circles!

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Suppose someone were to tell you that a vehicle moving at constant speed was accelerating. You would probably reply something to the effect that this could only happen if pigs could fly. But, listen, overhead... don't you hear "Oink, oink..."?

It turns out that a vehicle, moving at a constant speed, and travelling in a circle, is indeed accelerating. And, the physics behind this phenomenon is the basis of calculations for motor vehicles that enter into yaw.

In this *Page from a Physicist's Notebook,* we will take a look at the physics involved. There's nothing new to learn. We just need to apply what we already know to the situation of a vehicle travelling in a circle. We will then use physics and mathematics obtained from first principles to derive the equations that can be applied to motor vehicle crashes involving yaw.

The basic information that we need is shown in the side bar at the right of the page. So, let's see how this applies to – going round in circles!

But first, let's consider the difference between speed and velocity. Many police officers will state that velocity (V) is measured in m/s, and speed (S) is measured in km/h. Furthermore, it is believed that this is the difference between the two quantities.

From basic principles…

Equations of uniform motion

$v = v_0 + at$

Newton's laws of motion

$F = ma$

$W = mg$

Coefficient of friction:

$\mu = \frac{F}{W}$

Acceleration due to gravity

$g = 9.81\ m/s^2$

Similar triangles:

$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$

Trigonometry:

$\tan \Theta = \frac{Opposite}{Adjacent}$

It's a kind of traffic mythology. But, for our present purposes, we need to dig a little deeper – and explore the real relationship between speed and velocity.

Once again, we already know the answer. And, it's not really wrong to indicate that velocity is measured in m/s, and speed is measured in km/h; they very frequently are.

But, there is a more fundamental difference which is very important to the concept of yaw.

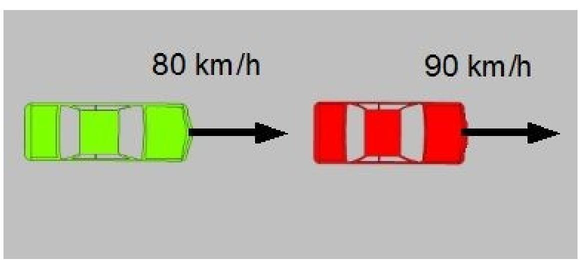
Speed and Velocity

In previous courses on the physics of collisions, I have frequently asked the question:

Supposing that a vehicle is travelling at 80 km/h, and a second vehicle is travelling at 90 km/h, what kind of a crash is going to occur?

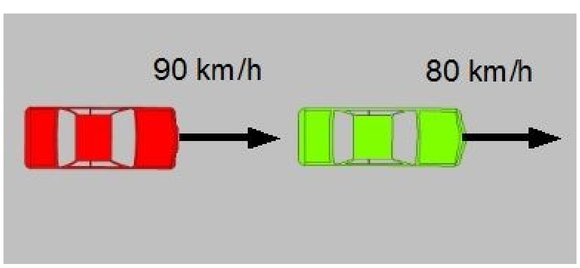
The usual answer is that the collision is going to be very severe. Two vehicles, each travelling at highway speed. Horrendous crash!

Then, I draw the following collision scenario:

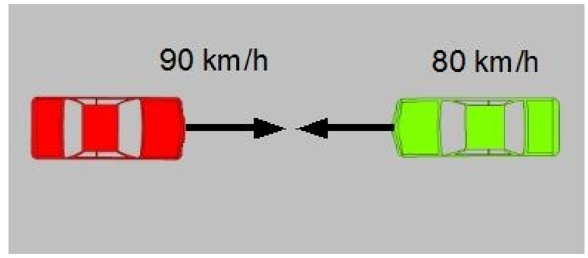


Hmmh. Not much of crash!

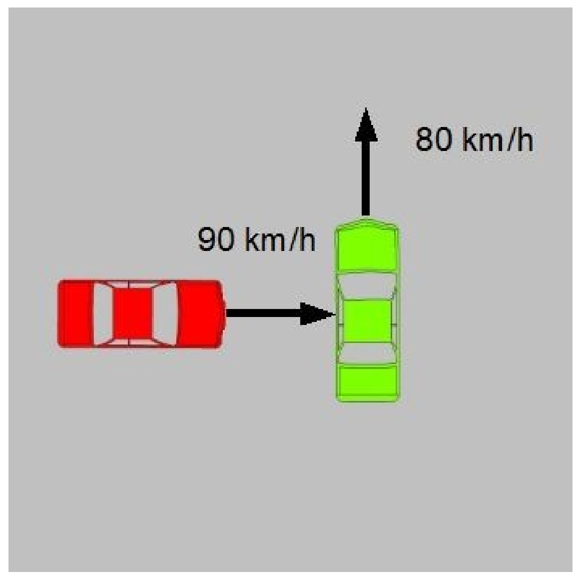
Nor is the next one:



Of course, when you think of a collision between cars travelling at 80 and 90 km/h, the image that first comes to mind is:



And, this head-on impact would indeed be a very severe collision. But, another possibility is the following:



This is also a crash between two vehicles travelling at 80 and 90 km/h and, as a side impact, at the speed of the bullet vehicle, this would also be very severe.

The common thread in all these scenarios is the initial speeds of the two vehicles – at 80 km/h and 90 km/h. But, even so, the crash outcomes are quite different.

Clearly, there is another very important factor at work. Direction! Vehicles travelling at a given (constant) speed can be moving in quite different directions. And, direction is extremely important in determining how a collision might – or might not! – occur.

So, the other piece of physics that we already know (but almost never consciously apply) is that the velocity of a vehicle is a vector quantity. It has a magnitude (which we term "speed"), and an associated direction (normally the vehicle's heading).

We can measure speed in km/h. In fact, these are the units we commonly use. They are used on the vehicle's speedometer, and , posted on highway signs. But, our American friends use miles per hour (mph) so, obviously, other units are possible for the quantity speed. The crews of ships and aircraft use nautical miles per hour (knots).

In fact, we can use whatever units we choose since the basic definition of speed is the rate of change of distance with time, or the distance travelled divided by the time taken ($v = d/t$). Any units for distance could be combined with any units for time but, conventionally, only certain combinations that give manageable numbers (e.g. tens of km/h) are commonly adopted.

So, while the magnitude of velocity is usually indicated in m/s, in principle, there is no reason it can't be defined as a (different) number of km/h. It isn't the units that define speed or velocity; it's whether or not we include the direction of travel. Velocity is speed in a specified direction. Speed is a scalar quantity (it's the magnitude or size of the velocity), while velocity is a vector.

Motion in a Circle

So, now let's consider what happens when a vehicle travels in a circle.

Clearly, the driver can apply a constant amount of throttle and so maintain a constant speed. Thus, we can definitely think about a vehicle travelling at constant speed (any value you like – say 13.9 m/s – 50 km/h). But, what happens to the vehicle's velocity as it travels around the circle?

Let's say that, at point A (Figure 1), the vehicle is travelling at 50 km/h due east. A short time later, at point B, it will be still be travelling at 50 km/h, but now it will be heading southeast. At point C it will be travelling at 50 km/h due south, and so on.

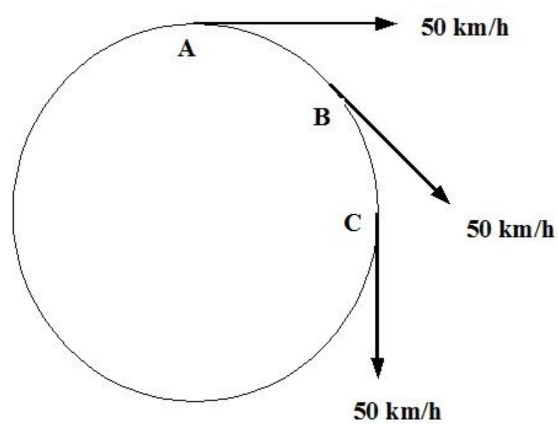


Figure 1. Motion in a circle at constant speed

The vehicle is travelling at constant speed, but its velocity is constantly changing – because the vehicle's heading is constantly changing as it moves around the circle.

Now, if the vehicle experiences a change in velocity, say in the time it takes to move between points A and B, it must also experience an acceleration, since acceleration is defined as the rate of change of velocity.

The definition of acceleration is the first of the equations of uniform motion:

$$v = v_0 + at \qquad (1)$$

Re-arranging this equation gives:

$[- v_0]$

$$v - v_0 = at$$

$[/t]$

$$a = \frac{v - v_0}{t} \qquad (2)$$

(Acceleration = change in velocity/time)

So, our vehicle that is moving in a circle, is travelling at constant speed, but, because its heading is changing over time, its velocity is changing, and the vehicle is accelerating.

Newton's First Law of Motion tells us that a vehicle will remain at rest or at constant speed in a straight line (constant velocity) unless acted upon by an unbalanced force. The other way of looking at this is that if the vehicle's velocity changes (i.e. the vehicle accelerates), there must be a force acting to cause the acceleration.

So, there must be a force acting on the vehicle that causes it to keep moving around the circle (as opposed to moving off at the constant speed of 50 km/h, in a straight line, which would be a tangent to the circle). This force is the side force (friction) acting between the road and the vehicle's tires as the tires try to slip sideways across the road (action) and the road surface pushes back against the tires (reaction).

Thus, the side force on the vehicle's tires produces the lateral acceleration that the vehicle experiences as it rounds the curve.

Newton's Second Law of Motion allows us to quantify the force:

$$F = ma$$

where:

F = Side force
m = Mass of the vehicle
a = Lateral acceleration

And, as suggested earlier, the lateral forces acting – the tire pressing against the road, and the road pushing back against the tire – are examples of action and reaction forces (Newton's Third Law of Motion – Action and reaction are equal and opposite.)

Note also, that the physical evidence on the vehicle and at the collision scene show the presence of these forces. The tire tread and sidewall can exhibit scuffing from the force applied by the rough roadway surface. “Yaw marks” on an asphalt pavement, or gravel thrown to the outside of the path of travel, are evidence of the vehicle's tires pressing against the road surface.

So, the physics that we already know – the equations for uniform motion and Newton's laws of motion – are entirely consistent with a vehicle moving at constant speed in a circle, requiring a lateral force to constantly change the vehicle's heading, and hence its velocity, and resulting in the development of lateral acceleration. The vehicle does indeed travel at constant speed and accelerate! It must!

Lateral Acceleration

In order to quantify a vehicle's motion around a curve (part of a circle), we first need to develop an equation for the lateral acceleration that is developed.

The rigorous mathematical derivation of this acceleration uses a technique known as (differential and integral) calculus. But, this subject is quite complex, and forms a course in itself, so we will resort to an approximation technique, that is more-or-less the basis used in calculus.

Figure 2 shows a vehicle rounding a curve. We are going to concentrate on the motion of the vehicle between points A and B.

The vehicle is travelling at constant speed such that:

$$v_1 = v_2 \qquad (3)$$

This equation tells us that the magnitude of the vehicle's velocity at point A is equal to the magnitude of the vehicle's velocity at point B (remember, that's what we mean by "speed").

But, the <u>velocity</u> of the vehicle at point A is different from the <u>velocity</u> of the vehicle at point B since the heading of the vehicle has changed between the two locations.

Recall that velocity is a vector. It has both magnitude (speed), and direction. In Figure 1, we have represented the two vectors by straight lines drawn to (some) scale, with arrows indicating the velocity's direction. Note that the two arrows are the same length, showing that the speed is constant, but point in different directions, showing the changing heading.

Vector Algebra

The graphical representation of vectors, and the mechanism by which they may be combined (vector triangle or parallelogram) is used extensively in calculations involving momentum. A full discussion of this topic, including the basic concepts of vector algebra, the derivation of the equation for conservation of linear momentum, plus graphical and algebraic solution techniques, have been discussed in previous *Pages From a Physicist's Notebook*. [1-3]

Since the vehicle's velocity has changed between points A and B, we can use a vector triangle (or parallelogram) to determine the actual change in velocity.

The relevant vector diagram is shown in Figure 3. The vector DE represents the vehicle's velocity at point A, while the vector DF represents the velocity at point B.

Now if:

$$\vec{\mathbf{DE}} = \vec{\mathbf{v_1}}$$

then:

$$\vec{\mathbf{DG}} = -\vec{\mathbf{v_1}}$$

DG is equal to the vector $-v_1$ since it has the same length as vector DE (v_1), but is pointing in precisely the opposite direction.

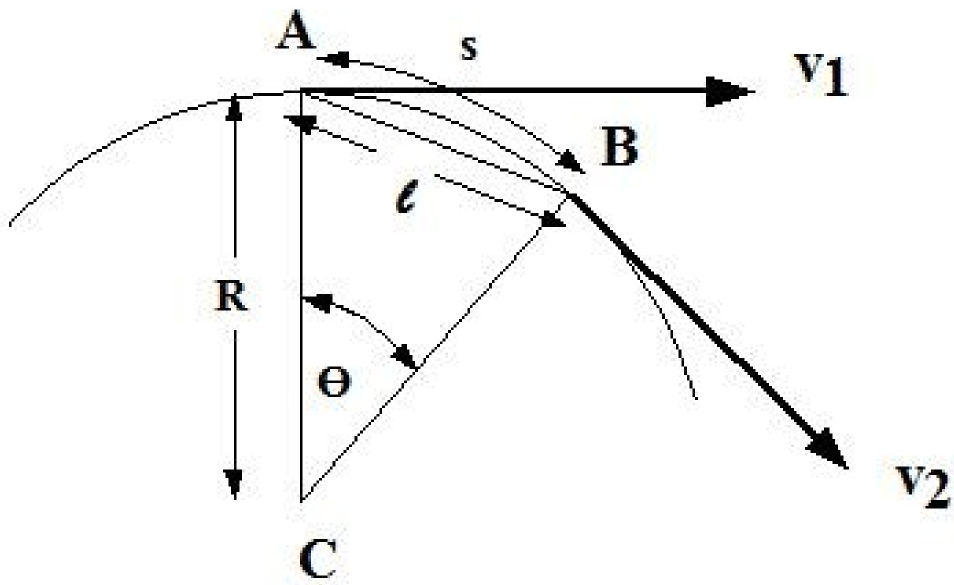


Figure 2. Motion in a circular arc

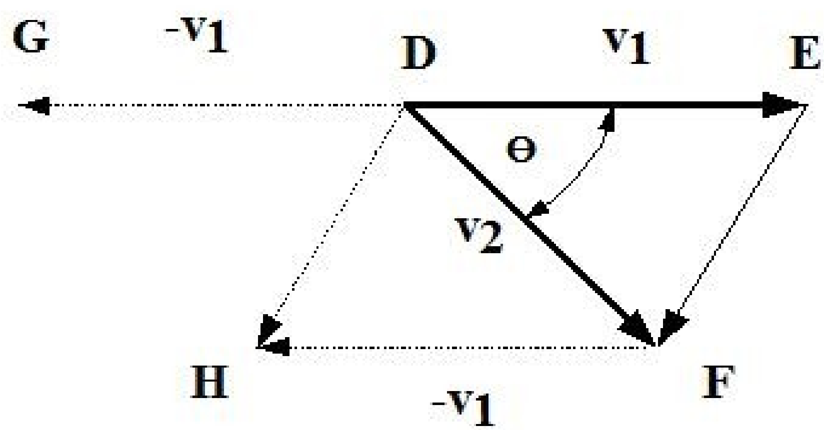


Figure 3. Vector Diagram for Vehicle Velocity Change

Note also that the vector FH is equal to vector DG since it has the same length (magnitude) and points in the same direction. So, in vector triangle DFH, we can see that:

$$\overrightarrow{\mathbf{DF}} + \overrightarrow{\mathbf{FH}} = \overrightarrow{\mathbf{DH}}$$

(DH is the resultant of the two vectors DF and FH.)

This means that:

$$\overrightarrow{\mathbf{v_2}} + (-\overrightarrow{\mathbf{v_1}}) = \overrightarrow{\mathbf{v_2}} - \overrightarrow{\mathbf{v_1}} = \overrightarrow{\mathbf{DH}}$$

$$\overrightarrow{\mathbf{DH}} = \overrightarrow{\mathbf{v_2}} - \overrightarrow{\mathbf{v_1}}$$

So, vector DH represents the vehicle's change in velocity $(v_2 - v_1)$ between points A and B along the circular arc.

We can also see from Figure 3, that the vector EF is the same as vector DH since it has the same length as DH and points in the same direction. Thus the vector EF also represents the vehicle's change in velocity between points A and B.

So, in vector triangle DEF, the vector DF represents the final velocity of the vehicle at point B (v_2), vector DE represents the initial velocity of the vehicle at point A (v_1), and vector EF represents the change in velocity $(v_2 - v_1)$.

Now, comparing Figures 2 and 3, we can see that DEF and CAB are similar triangles. They have the same angles, so that the sides of the triangles are in the following proportions:

$$\frac{AB}{AC} = \frac{EF}{DE}$$

Now, AB is the linear distance (ℓ) between points A and B, while AC is the radius (R) of the circle. Also, the length of DE is the change in vehicle speed $(v_2 - v_1)$ and the length of EF is the vehicle's speed (v_1).

Consequently, we have:

$$\frac{\ell}{R} \quad \frac{(v_2 - v_1)}{v_1}$$

[x v_1]

$$(v_2 - v_1) = \frac{\ell}{R} v_1$$

Now, let's divide each side of the equation by t, the time taken for the vehicle to travel between points A and B (and hence the time taken to change its velocity from v_1 to v_2).

[/ t]

$$\frac{(v_2 - v_1)}{t} = \frac{\ell}{R\,t} v_1$$

But, we know from one of the equations for uniform motion (Equation 2) that, in the present case, $(v_2 - v_1)/t$ is the vehicle's lateral acceleration.

Consequently:

$$a = \frac{\ell}{R\,t} v_1 \qquad (4)$$

We also know that the vehicle is travelling at constant speed (V) so that we can write Equation 3 as:

$$V = v_1 = v_2$$

This gives Equation 4 as:

$$a = \frac{\ell}{R\,t} V \qquad (5)$$

So far, we have an expression for the magnitude of the lateral acceleration in terms of the constant vehicle speed (V), the radius of the circle (R), the distance (ℓ) and time (t) for the vehicle to move between points A and B.

Note that, in Figure 3, the actual distance travelled by the vehicle between points A and B is the length of the arc (s). Since the vehicle is moving at constant speed, the distance travelled (s) will be given by:

$$s = V\,t \qquad (6)$$

Now it's time to do a little approximating. Suppose that points A and B are very close together. The angle Θ will be very small, and the distance along the arc (s) will almost be the same as the linear distance (ℓ) between points A and B. Consequently, we can write Equation 6 as:

$$\ell \approx V\,t$$

And, in the limit as Θ approaches zero (this is the infinitesimal calculus bit!):

$$\ell = V\,t \qquad (7)$$

so that:

[/ v]

$$t = \ell / V \qquad (8)$$

We can now substitute for the time, t, from Equation 8 into Equation 5 to give:

$$a = \frac{\ell V}{R\,(\ell / V)}$$

$$a = \frac{V^2}{R} \qquad (9)$$

Equation 9 gives the magnitude of the lateral acceleration. But, since acceleration is a vector quantity, we should also consider its direction. Going back to our approximation, as the angle Θ gets very small and approaches zero, the line BC will almost be the same as the line AC. Similarly, the vectors ED and EF will almost be the same. In the limit, as Θ goes to zero, the change in velocity (vector EF) will act at right angles to the vehicle's velocity (vector ED), and hence be pointing towards the centre of the circle. Since the acceleration vector must act along the same line as the change in velocity [$a=(v_2 - v_1)/t$], it is evident that the vehicle's acceleration must be towards the centre of the circle.

Since the acceleration of the vehicle is acting at right angles to the vehicle's forward (longitudinal) velocity, we term the acceleration the lateral acceleration.

Motion in a circle

Lateral acceleration, $a = \frac{v^2}{R}$

Yaw Equation

Now that we have an expression for a vehicle's lateral acceleration as it rounds a curve, we can consider a way to quantify this acceleration.

As we noted earlier, the presence of the lateral acceleration means that there is a lateral force acting on the vehicle, constantly causing the change in heading to produce the circular motion. As the tires try to slip sideways across the road surface, the road pushes back against the tires so that friction provides the necessary side force.

Newton's Second Law of Motion (F=ma), tells us how large this force has to be for a given vehicle mass (m) and lateral acceleration (a).

Thus, the lateral force is given by:

$$F = m\,a$$

From Equation 9, we know that the lateral acceleration is V^2/r so substituting this for a in the above equation gives:

$$F = m\,\frac{V^2}{R} \qquad (10)$$

We also know that the maximum frictional force (F) that can be produced is dependent on the weight of the vehicle (W=mg) and the available coefficient of friction between the tires and the road surface (μ) where

$$F = \mu W = \mu mg \qquad (11)$$

Substituting for F from Equation 11 into Equation 10 gives:

$$\mu mg = m\,\frac{V^2}{R}$$

[/ m]

$$\mu g = \frac{V^2}{R}$$

[x R]

$$V^2 = \mu\,g\,R$$

But, g = 9.81 m/s², so that:

$$V^2 = 9.81\,\mu\,R$$

[√]

$$V = \sqrt{(9.81\,\mu\,R)}$$

$$V = 3.13\sqrt{(\mu\,R)} \qquad (12)$$

Since the gravitational acceleration (g) is measured in m/s^2, and we usually measure the radius of the curve (R) in m, the above equation gives the speed (V) in m/s.

To convert the speed in m/s to km/h, we use the fact that 1 m/s = 3.6 km/h, so that:

$$S = 11.28\sqrt{(\mu\,R)} \qquad (13)$$

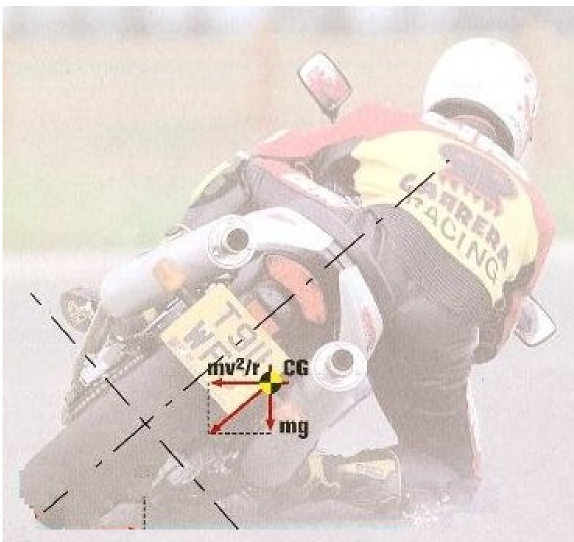
Yaw Equation

$$\mathbf{V = 3.13\sqrt{(\mu\,R)}\ \ m/s}$$

$$\mathbf{S = 11.28\sqrt{(\mu\,R)}\ \ km/h}$$

Lean Angle

Motorcyclists generate lots of side force by leaning into the turn. For example, think of motorcycle racing, where the rider leans the machine into the curve, but also hangs out as far as possible, scraping his leathers against the track! The rider generates the maximum side force by creating the biggest lean angle – without falling over!



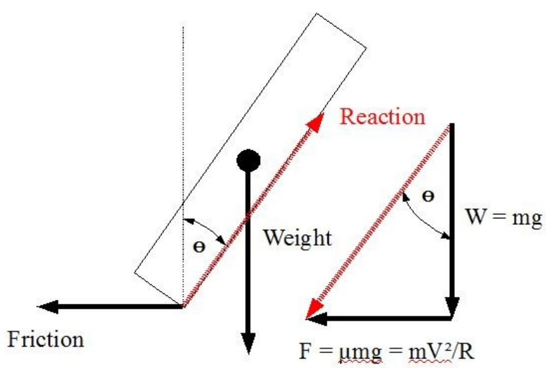


Figure 4. Motorcycle Lean Angle

We can see how the force is generated by looking at a sectional view of the motorcycle rounding a level curve (Figure 4).

In the simplified diagram, we can see that, for a given speed (V) around a curve of constant radius (R), the lean angle (Θ) is determined by the ratio of the frictional side force (F) and the weight of the motorcycle and rider (W) . In particular, the triangle of forces gives:

$$\tan \Theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{F}{W} = \frac{\mu mg}{mg}$$

Consequently:

$$\tan \Theta = \mu \qquad (14)$$

Motorcycle Lean Angle

$$\Theta = \tan^{-1}(\mu)$$

References

1. German A; Page From a Physicist’s Notebook; Momentum 101 - The principle of conservation of linear momentum
2. German A; Page From a Physicist’s Notebook; Momentum 102 - Vector Analysis and Momentum
3. German A; Page From a Physicist’s Notebook; Momentum 103 - The algebraic solution for conservation of momentum

Author

Alan German is a research physicist who obtained both a BSc and a PhD from the University of Salford in the United Kingdom. For over 30 years he has been involved in the study of motor vehicle safety through a programme of in-depth investigations of real-world crashes.

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His research interests include the collision performance of occupant restraint systems, collision investigation and reconstruction techniques, and the application of microcomputers to motor vehicle safety. He has authored or co-authored over eighty publications on a variety of aspects of traffic safety. He was co-author of the paper "The Use of Event Data Recorders in the Analysis of Real-World Crashes" that received the inaugural Dr. Charles H. Miller Award for the best technical paper presented at the 12th. Canadian Multidisciplinary Road Safety Conference.

In 2001 Alan was presented with a U.S. Government Award for Safety Engineering Excellence, and in 2006 was the recipient of a President's Choice Award from the Canadian Association of Technical Accident Investigators and Reconstructionists.