Page From a Physicist's Notebook

Work is mad - it takes energy!

Alan German, PhD CPhys
Collision Investigation and Research
Transport Canada

One of my long-standing issues with most reconstruction courses offered to police is the "cookbook" approach. Essentially, instructors hand out a manual with pages (and pages) of equations, and then go on to explain the parameters in these equations and how to calculate the result. Now, cookbook physics is fine - as far as it goes. The difficulty is that, without some understanding of the physical basis for an equation, a method may well be applied in an inappropriate situation. Worse, the user is limited to just the equations in the book, and hence may be unable to find "the one" (or, more likely, "the series" of equations) necessary for a particular sequence of collision events. Also, some individuals have been criticized by the courts for being unable to explain the basis for a calculation method. So, the intent here is to try to provide the basic physics (and mathematics) behind some of the formulae routinely used by collision reconstructionists.

In this page from a physicist's notebook we will explore the basis for the slide-to-stop equation. To do so, we will use familiar equations and concepts, combining these and following through the mathematics to provide a rigorous derivation of the resultant equations. In the box at the right, the basic equations, their source, and the well-known physical principles that we are going to use are listed. So, to battle!

From basic principles…

Equations of uniform motion
$2ad = v^2 - v_0^2$

Newton's laws of motion
$F = ma$
$W = mg$

Coefficient of friction
$\mu = F/W$

Principle of mechanical work
$Work = Fd$

Principle of conservation of energy
Work done = Kinetic energy change

Acceleration due to gravity
$g = 9.81\ m/s^2$

Kinetic Energy

If a vehicle is accelerated from a stop, over some distance d, it will then be moving at a (final) velocity of v. One of the equations for uniform motion tells us that the vehicle's acceleration (a), its distance travelled (d), and the final velocity (v) are related by the equation:

$$2ad = v^2 \qquad (1)$$

(From $2ad = v^2 - v_0^2$, where in this specific case the vehicle's initial velocity is zero, $v_0=0$).

Now, comes one of the world's great mysteries (soon to be followed by an even greater puzzle.) For some, apparently inexplicable, reason we now decide to use the fact that we can maintain the equality expressed in equation (1) if we divide each side by 2. For now, let's not worry about why we would do such a thing, let's just be content that dividing both sides of the equation by 2 is perfectly sound from a mathematical perspective. So, dividing each side of equation (1) by 2 gives:

$$ad = \tfrac{1}{2} v^2 \qquad (2)$$

Now, let's go completely wild and multiply both sides of equation (2) by m:

$$mad = \tfrac{1}{2} mv^2 \qquad (3)$$

This seems like a strange thing to do - slightly mad in fact! However, let's recall that our result so far is purely based on one of the equations for uniform motion and simple mathematical operations that have maintained the stated equality.

Looking at the left side of equation (3) we can see that ma (the vehicle's mass times its acceleration) is an expression for the tractive force (F) that was applied to the vehicle to cause it to accelerate. Newton's second law of motion (F=ma) tells us that instead of ma we can write F. Thus, equation (3) becomes:

$$Fd = \tfrac{1}{2} mv^2 \qquad (4)$$

Now we realize that the left side of this equation, the force acting on the vehicle multiplied by the distance travelled, is an expression for the work done in accelerating the vehicle. Thus, equation (4) says that:

$$\text{Work done} = \tfrac{1}{2} mv^2 \qquad (5)$$

Applying the principle of conservation of energy tells us that the work done on the vehicle shows up as a gain in the vehicle's energy. Because this is energy associated with the motion of the vehicle, it is termed kinetic energy [Greek: kineo, to move].

$$\text{Gain in kinetic energy} = \tfrac{1}{2} mv^2 \qquad (6)$$

When the vehicle was initially stopped, it had no kinetic energy (it wasn't moving). By applying a tractive force (F) over a distance (d), an amount of work is converted into kinetic energy of $\tfrac{1}{2} mv^2$. Thus, when a vehicle of mass m is moving at a velocity v it has kinetic energy (KE) of $\tfrac{1}{2} mv^2$.

$$KE = \tfrac{1}{2} mv^2 \qquad (7)$$

This is an important result because it is the basis of all collision reconstruction equations that use energy. (Perhaps we weren't so mad after all?)

We have derived the expression for kinetic energy: KE = ½ mv²

Slide to Stop

Probably the most important of such equations is that for a vehicle, initially travelling at some velocity v, being braked hard to a stop. We assume full braking so that friction between the locked wheels and the road provides the resistive force.

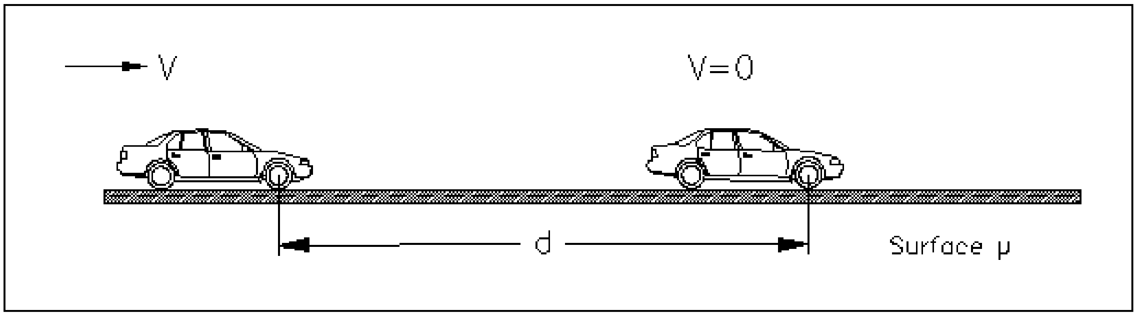

Figure 1. Slide to a stop

In Figure 1, we have a vehicle, initially travelling at speed v, in full braking over a stopping distance d on a surface with a coefficient of friction of μ. Applying the principle of conservation of energy to this situation gives that:

Loss in kinetic energy =
Work done by the force due to friction

$$\text{Initial KE} = \tfrac{1}{2} mv^2 \qquad (8)$$

$$\text{Final KE} = 0 \qquad (9)$$

It follows that:

$$\text{Loss in kinetic energy} = \tfrac{1}{2} mv^2 - 0$$

$$\text{Loss in kinetic energy} = \tfrac{1}{2} mv^2 \qquad (10)$$

Work done by the frictional force (F) is the product of this force and the vehicle's stopping distance (d).

$$\text{Work done} = Fd \qquad (11)$$

The coefficient of friction (μ) is the ratio of the frictional force (F) and the weight of vehicle (W):

$$\mu = F/W \qquad (12)$$

Thus, the frictional force is given by:

$$F = \mu W \qquad (13)$$

Since, as a special case of Newton's second law of motion, the force of gravity acting on the vehicle, the vehicle's weight (W), is equal to the mass of the vehicle (m) times the gravitational acceleration (g):

$$W = mg \qquad (14)$$

Substituting for W from equation (14) in equation (13) we have:

$$F = \mu mg \qquad (15)$$

And, substituting for F from equation (15) in equation (11) we have:

$$\text{Work done} = Fd = \mu mg\, d \qquad (16)$$

We can now apply the principle of conservation of energy for the vehicle in braking to a stop such that the loss in kinetic energy given by equation (10) is equal to the work done by the force due to friction given by equation (16). Thus, we can write:

$$\text{Loss in kinetic energy} = \text{Work done}$$

$$\tfrac{1}{2} mv^2 = \mu mgd \qquad (17)$$

Solving for the vehicle speed, based on the measured stopping distance and coefficient of friction gives:

$$v^2 = \frac{2}{m}\mu mgd = 2\mu gd$$

$$v = \sqrt{2\mu gd}$$

We now apply the fact that the acceleration due to gravity, $g = 9.81\ m/s^2$ so that:

$$v = \sqrt{2\mu 9.81d} = \sqrt{19.62\mu d} = 4.43\sqrt{\mu d}$$

$$v = 4.43\sqrt{\mu d} \qquad (18)$$

Note that the above equation gives the vehicle speed in base units of m/s since the stopping distance is measured in m and the gravitational acceleration is measured in metres per second squared.

Keep it simple – use basic units

The use of such units in basic calculations is to be encouraged, especially when dealing with complex situations where multiple stages of vehicle energy dissipation are being considered. One must take great care not to mix speeds measured in kilometres per hour with relationships, using distances in metres and gravitational acceleration in metres per second squared, that are effectively considering speeds in m/s.

However, normally, at the end of a calculation sequence, we wish to provide a speed estimate in kilometres per hour since these units are readily understood by everyone.

Conversion factor m/s to km/h…

1 km/h = 1000 m / 60 x 60 s
= 1000/3600 m/s
= 1/3.6 m/s

Consequently:

$$1\ m/s = 3.6\ km/h \qquad (19)$$

Multiplying any result in m/s by 3.6 gives the result in km/h.

Note that the conversion factor of 3.6 is easy to remember because it is based on there being 3600 seconds in an hour and 1000 metres in a kilometre.

The inverse factor, 1/3.6 results in a recurring decimal value (0.27777…) which needs to be rounded for use in calculations.

So, we need to apply a conversion factor to switch between m/s and km/h.

Using the conversion factor from equation (19), in our expression for vehicle speed in equation (18) gives:

$$v = 3.6 \times 4.43\sqrt{\mu d}$$
$$v = 15.9\sqrt{\mu d} \quad km/h \qquad (20)$$

This is the simple form of the equation for locked wheel braking to a stop. The calculation methodology is a little more complex if the vehicle is travelling up or down a grade, or if not all the vehicle's wheels are locked. But, that's another page (or two) in the physicist's notebook.

Change in Kinetic Energy

At this point we might want to go back to equation (17) and note that this expression is a special case of a loss in vehicle kinetic energy due to work done against the force of friction under braking. The special situation is that the vehicle comes to a stop such that the final kinetic energy is zero. In a more general case, the vehicle may be braked from a high speed (v_1) to a lower speed (v_2), in which case we could write:

Loss in kinetic energy =
Work done by the force due to friction

$$\tfrac{1}{2} mv_1^2 - \tfrac{1}{2} mv_2^2 = \mu mgd \qquad (21)$$

Note that, while this equation doesn't appear in the books of formulae normally provided to police officers, it is an exceedingly useful formulation. It is a perfectly general expression for computing the kinetic energy lost when a vehicle is travelling over a surface with a given coefficient of friction. Any sequence of vehicle decelerations over multiple surfaces with different frictional properties can be treated by handling the motion over each surface using this equation. More importantly, individual calculations for such sequences can be combined in order to determine the initial speed of a vehicle in such circumstances.

Braking over Multiple Surfaces

For example, as shown in Figure 2, consider a vehicle that slides to a stop by initially travelling a distance d_1 over a surface with a coefficient of friction of μ_1 and then, subsequently, travels a distance d_2 over a surface with a coefficient of friction of μ_2. Let's assign v_1 as the initial speed of the vehicle as it enters onto the first surface, and v_2 as the speed of the vehicle at the boundary of the two surfaces.

The trick in all these more complex situations is to start simply, that is begin from the point at which the vehicle came to rest, and work backwards along the path of travel until we compute the vehicle's initial speed. We can do this in a series of individual calculation steps, in which case we can handle just about any series of events. Or, since most real-world situations are not that complex, we can derive a formula that will enable us to calculate the vehicle's initial speed from just the measured stopping distances and the coefficients of frictions for the two surfaces.

For the final portion of the vehicle's path of travel, which effectively is a slide to stop from an initial speed of v_2, we can use equation (17) in the form:

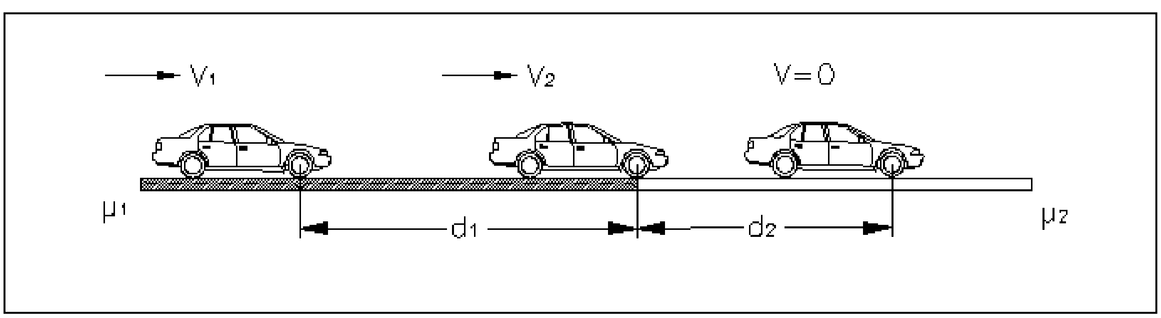

Figure 2. Vehicle braking to a stop over two different surfaces

$$\tfrac{1}{2} m v_2^2 = \mu_2 m g d_2 \qquad (22)$$

For the vehicle's path of travel over the first surface, we can use equation (21) in the form:

$$\tfrac{1}{2} m v_1^2 - \tfrac{1}{2} m v_2^2 = \mu_1 m g d_1 \qquad (23)$$

Now, in the above equation, we can substitute for $\tfrac{1}{2} m v_2^2$ from equation (22) to give:

$$\tfrac{1}{2} m v_1^2 - \mu_2 m g d_2 = \mu_1 m g d_1 \qquad (24)$$

Re-arranging equation 24 gives:

$$\tfrac{1}{2} m v_1^2 = \mu_1 m g d_1 + \mu_2 m g d_2 \qquad (25)$$

Simplifying this equation gives:

$$\begin{aligned} v_1^2 &= 2\,(\mu_1 g d_1 + \mu_2 g d_2) \\ &= 2g\,(\mu_1 d_1 + \mu_2 d_2) \\ &= 2 \times 9.81\,(\mu_1 d_1 + \mu_2 d_2) \\ &= 19.62\,(\mu_1 d_1 + \mu_2 d_2) \end{aligned}$$

Taking the square root gives:

$$v_1 = \sqrt{19.62(\mu_1 d_1 + \mu_2 d_2)}$$

$$v_1 = 4.43\sqrt{(\mu_1 d_1 + \mu_2 d_2)} \qquad (26)$$

Note that the above equation gives v_1 in m/s. To convert the speed to km/h we must multiple by 3.6 to give:

$$v_1 = 15.9\sqrt{(\mu_1 d_1 + \mu_2 d_2)} \ \text{ km/h} \qquad (27)$$

Note that equations (26) and (27) give the vehicle's initial speed in m/s and km/h respectively.

It is left for the reader as an exercise to determine the equivalent expression should three different surfaces be involved in the braked path. [Hint! One might recognize a pattern between equations (20) and (27).]

Combined Speed Formula

Another great mystery of science is the combined speed formula which is usually written as:

$$v^2 = v_1^2 + v_2^2 \qquad (28)$$

One application of this equation is for a slide to stop calculation over two different surfaces. The method is as follows:

Step 1 – Calculate v_2 as a slide to stop using equation (20), with the relevant coefficient of friction (μ_2) and stopping distance (d_2), with the result in km/h.

Step 2 – Calculate an "equivalent speed", v_1, as a slide to stop over the first surface using equation (20), with the coefficient of friction (μ_1) and stopping distance (d_1), with the result in km/h.

Step 3 – Calculate the vehicle's initial speed, v, using equation (28), the combined speed formula, giving the final result in km/h.

Take special note of two things. Firstly, the above uses different notation than we used in Figure 2. In equation (28), v is the vehicle's initial speed, that is the speed as the vehicle starts to brake on the first surface. v_2 is still the speed of the vehicle as it enters and brakes over the second surface. Secondly, v_1 is not a real speed at all. It is calculated as a slide to stop in a situation where the vehicle doesn't stop! In fact, on the first surface, the vehicle slows down from a speed of v to a speed of v_2. So, why does this method work?

The answer, of course, is that it depends on the magic of physics. So, let's analyze the situation using basic physical principles and see how to derive the combined speed formula.

For the vehicle braking over the second surface, equation (22) applies:

$$\tfrac{1}{2} mv_2^2 = \mu_2 mgd_2 \qquad (22)$$

This equation will allow us, in Step 1, to calculate, v_2, the speed of the vehicle as it enters the second surface.

And, for the vehicle braking over the first surface, a version of equation (23) applies:

$$\tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_2^2 = \mu_1 mg\, d_1 \qquad (23)$$

Note that the first two terms in the above equation represent the loss in kinetic energy of the vehicle under locked-wheel braking across the first surface. The initial vehicle speed is v, and the "final" vehicle speed (at the end of the first surface) is v_2.

This is also precisely the amount of energy that would be dissipated by a vehicle, initially travelling at a speed v_1, that braked to a stop on the first surface. Thus we can write:

$$\tfrac{1}{2}mv_1^2 = \mu_1 mg\, d_1 \qquad (29)$$

The above equation is used in Step 2 to calculate the value of v_1.

Now, we can substitute for $\mu_1 mg\, d_1$ from equation (29) into equation (23) to give:

$$\tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_2^2 = \mu_1 mg\, d_1 = \tfrac{1}{2}mv_1^2$$

Hence:

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_2^2 = \tfrac{1}{2}mv_1^2 \qquad (30)$$

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2$$

$$v^2 = v_1^2 + v_2^2 \qquad (31)$$

The combined speed formula derived !

The "trick" to the method is that v_1 is not a real speed for the subject vehicle. It's effectively a measure of the kinetic energy dissipated by the vehicle while braking across the first surface, expressed as an equivalent slide-to-stop speed.

What we have really done is equated the actual loss in the vehicle's kinetic energy in braking over a distance d_1 on the first surface (which, from equation (23), is $\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_2^2$), to the kinetic energy lost by a vehicle braking to a stop over a distance d_1 on the first surface (which, from equation (8), is $\tfrac{1}{2}mv_1^2$). Both situations involve locked wheel braking and require work to be done, in the amount of $\mu_1 mg\, d_1$, by the force of friction acting over the braking distance.

The combined speed formula is really a statement of the principle of conservation of energy. The initial kinetic energy of the vehicle ($\tfrac{1}{2} mv^2$) is dissipated as the loss in kinetic energy across the first surface ($\tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_2^2$) plus the loss of kinetic energy across the second surface ($\tfrac{1}{2}mv_2^2$).

Mathematically, this would be expressed as:

$$\tfrac{1}{2}mv^2 = (\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_2^2) + \tfrac{1}{2}mv_2^2$$

Our "trick" is to express the first amount of kinetic energy loss as $\frac{1}{2}mv_1^2$ (equation 30) so that:

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2$$

and hence:

$$v^2 = v_1^2 + v_2^2 \qquad (31)$$

The science behind the combined speed formula is, therefore, the principle of conservation of energy

Rather than using the concept of combined "speeds", since the method is really that of conservation of energy, it would be preferable to apply equation (25) directly to the braking situation. Such basic principles can be adapted for the solution of any energy-based problem as illustrated below.

Example

Everyone knows how to use equations like (18) and (26) (or more likely (20) and 27)) when we have a vehicle undergoing multiple braking events. The intention of this article was not to discuss how to use the equations, but rather to provide some knowledge of the underlying basic principles of physics that can be applied in any appropriate situation. So, as an example of the basic method of handling vehicle energies over a sequence of collision events, let's use a slightly different and somewhat more general crash situation.

A 2000 Ford Taurus was travelling along a four-lane, median-divided urban roadway when the driver lost directional control. Heavy pre-impact brake marks ran from the driving lane, across the paved right shoulder, leading into an impact with a concrete bridge abutment. The posted speed limit was 60 km/h. The length of the tire marks was measured as 25.5 m, and the coefficient of friction for the asphalt pavement was determined to be 0.7. There was broad crush across the entire front end of the Taurus, measuring approximately 40 cm. The driver was unrestrained and sustained fatal injuries. The investigating coroner wishes to know if speed was a significant factor in this crash.

A frontal barrier crash test conducted by Transport Canada on a similar 2000 Ford Taurus produced between 38 and 42 cm of crush to the front end of the test vehicle. The staged collision was conducted at an impact speed of 47.8 km/h. Due the similarities in the vehicle damage profiles, we can estimate that the Taurus in the real world collision was travelling at approximately 48 km/h at the point at which it struck the bridge abutment.

Applying the principle of conservation of energy to the pre-impact braking, we can use equation (23):

$$\tfrac{1}{2}\,mv_1^2 - \tfrac{1}{2}\,mv_2^2 = \mu_1 mgd_1 \qquad (23)$$

where:

m = vehicle mass
v_1 = initial speed prior to the loss of control
v_2 = speed at the end of the tire marks (i.e. at the point of impact with the bridge)
μ_1 = coefficient of friction between the vehicle's tires and the roadway
g = gravitational acceleration (9.81 m/s²)
d_1 = measured braking distance

Note that equation (23) can be simplified to:

$$v_1^2 = 2\mu_1 gd_1 + v_2^2$$

In this equation, our measured and derived parameters are as follows:

$\mu_1 = 0.7$
$g = 9.81\ m/s^2$
$d_1 = 25.5\ m$
$v_2 = 48\ km/h = 13.4\ m/s$ (crash test)

It follows that:

$$v_1^2 = 2 \times 0.7 \times 9.81 \times 25.5 + 13.4^2$$

$$v_1^2 = 350.2 + 179.6 = 529.8$$

$$v_1 = \sqrt{529.8} = 23\ m/s$$

$$v_1 = 83\ km/h$$

Note that in performing the calculation using this form of the equation, we converted the barrier equivalent speed from the crash test (v_2) from km/h to m/s since the equation uses basic units (d_1 in m and g in m/s^2).

Note also that another way of looking at equation (23) in this collision situation is as an expression of the conservation of energy for the vehicle's motion. If we rearrange equation (23), we have:

$$\tfrac{1}{2} m v_1^2 = \mu_1 m g d_1 + \tfrac{1}{2} m v_2^2$$

The three terms in this equation indicate that the vehicle's initial kinetic energy ($\tfrac{1}{2} m v_1^2$) is dissipated as the work done against friction when the vehicle was braking on the asphalt pavement ($\mu_1 m g d_1$) and the energy dissipated in crushing the vehicle's front end when it struck the bridge abutment ($\tfrac{1}{2} m v_2^2$).

Thus our knowledge of the principle of conservation of energy, combined with the principle of mechanical work, allows us to perform a calculation that will assist in responding to the coroner's question.

Clearly, the case vehicle was travelling above the posted speed limit. Was driving at 83 km/h on a roadway with a 60 km/h limit a major factor in this crash? Perhaps we need to consider other factors related to the collision in order to know the whole story. While, being one piece of the puzzle, speed alone is not always the complete answer.

Acknowledgements

The assistance of Jamie White, Collision Investigation and Research Division, Transport Canada, in producing the figures used in this article is greatly appreciated.

Author

Dr. Alan German is a Chartered Physicist who has been involved in the in-depth investigations of real-world crashes for over 25 years. He has lectured on the physics of crashes on a number of collision reconstruction courses.

Alan is currently Chief of the Collision Investigation and Research Division, Road Safety and Motor Vehicle Regulation Directorate, Transport Canada.

He is a Past President, and a current member of the Board of Directors, of the Canadian Association of Road Safety Professionals, and has served as a member of the Board of Directors of the Association for the Advancement of Automotive Medicine. He is a charter member of both the Eastern Ontario Traffic Investigators Society (EOTIS) and of the Canadian Association of Technical Accident Investigators and Reconstructionists (CATAIR).