Page From a Physicist's Notebook

Momentum 102
Vector Analysis and Momentum

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In a previous *Page from a Physicist's Note-book* (*Momentum 101 – The principle of conservation of linear momentum*) [1] we saw how the application of Newton's laws of motion to a vehicle-to-vehicle crash resulted in the derivation of the equation for the conservation of linear momentum (see box at right).

It was also noted that, because momentum is the product of a scalar quantity (vehicle mass) and a vector quantity (vehicle velocity), momentum itself is a vector. And, since both the magnitude (size) and direction of vector quantities are important, we also discovered that working with this equation required the application of vector analysis rather than simple algebra.

Anyone who was still awake after reading through Momentum 101 may also have realized that we have a seemingly strange situation – one equation, with two unknowns. We can readily determine the masses of the two vehicles (m_1 and m_2), and we can usually estimate the two separation velocities (V_1' and V_2'). But, the initial velocities of both vehicles (V_1 and V_2) are normally unknown. We generally want to know the vehicle speeds! So, how is it possible to solve a single equation when two parameters are unknown?

From basic principles…

Conservation of linear momentum

$$m_1\vec{V_1} + m_2\vec{V_2} = m_1\vec{V_1'} + m_2\vec{V_2'}$$

The answer is that this is a vector equation, and we can look at the conservation of momentum in two perpendicular directions (x and y) which effectively gives us two equations and two unknowns that can be solved simultaneously.

But, there's no free lunch here. While we only have to deal with a single equation, we do have to work in two dimensions, and will need to provide a bunch of data from the crash to identify the vehicle run-out speeds, and the approach and departure angles of both vehicles into and out of the collision.

In this page from a physicist's note-book, we will look at the properties of vectors, how to add them together to determine a resultant vector, and, in particular, how to find the initial speed of each of two vehicles involved in a vehicle-to-vehicle crash.

Vectors and Scalars

We know that vectors have two important properties – magnitude (size) and direction. In the case of velocity, we refer to the magnitude of this vector as speed, and we often specify the associated direction in terms of the compass points. For example, the classic first sentence of a motor vehicle accident report starts something like: "Vehicle 1 was northbound at 50 km/h." Note that both the speed (50 km/h), and the direction (north), of the vehicle have been indicated so that the vehicle's velocity has been fully specified.

[Note also that we can specify the same velocity as 13.9 m/s in a northerly direction. It's the combination of the speed, and the direction, that defines a vector quantity, not the units used to measure speed, which is the vector's magnitude.]

Magnitude and direction specify a vector (not units !)

A variety of other collision-related parameters can be seen to be vectors. For example, collision force may be applied to the front of a vehicle and cause it to slow down. Conversely, the force may be applied to the rear of the vehicle and cause it to speed up. Similarly, we think of acceleration taking place in a forward direction when a vehicle speeds up, or rearwards (deceleration) when the vehicle slows down. Since direction is obviously important for considering the nature of both force and acceleration, in addition to how great a quantity of each is involved in a given situation, it is clear that both force and acceleration are vector quantities.

Conversely, scalars such as vehicle mass have no associated direction. For example the 1150 kg mass (curb weight *) of a 1998 Ford Escort station wagon merely tells us how much material (steel, glass, plastic, fuel, oil, and coolant) was delivered to the vehicle's purchaser. The total mass of a collision-involved vehicle will include this original mass, plus the mass of any occupant(s), and that of any cargo. But, we are still only considering the total amount of material (of various types) comprising one of the objects involved in the crash. There is no direction associated with this "lump" of matter.

Vector Analysis

The equation for the conservation of linear momentum tells us that the sum of the vehicle momenta before the crash is equal to the sum of the momenta after the crash. Clearly, in order to work with this equation, we need to know how to add two momentum vectors together. And, we also need to know how to represent momentum, as the product of a mass and a velocity, in a vector format.

While these requirements sound complicated, it turns out that we already know how to do both. We just don't think of a commonly used process as an exercise in vector analysis. But we will in a few moments!

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* Note that, in this context, "curb weight" is a misnomer. Weight is a force; it's the gravitational attraction of the earth acting on the vehicle, and thus should be expressed in Newtons (N) rather than kilograms. However, curb weight is the term in common usage for the more scientifically accurate term of mass.

Suppose that we were to get into our car and drive 30 km due east. Then, let's suppose that we turn left and drive for a further 40 km due north. We will have travelled a total distance of 70 km but, as the crow flies, we are only 50 km from home. We can readily see that this is the case if we consult our map (Figure 1).

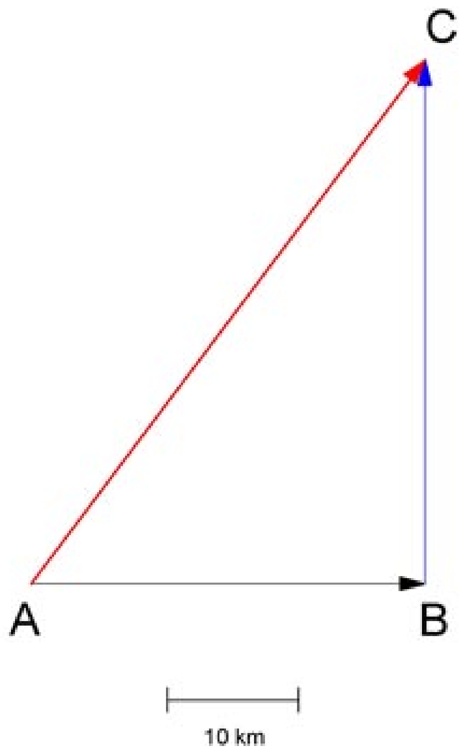


Figure 1 Vector Triangle

Home was at point A. Initially we travelled 30 km east to point B, and then 40 km north to arrive at point C. Effectively, we have travelled 50 km to the east of north and, had there been a road in this specific direction, it's clear that we could have travelled directly from point A to point C.

Note that we keep specifying distances travelled in given directions. Distance is a scalar quantity, measured in kilometres (or metres if we prefer). If we associate a particular distance (a scalar), with a specific direction, we have specified a vector quantity, this vector being known as displacement. In our two-dimensional world (for our purposes, we can ignore the vertical direction), displacement is a measure of the absolute change in position between an origin and a destination.

So, our mapping exercise shows us how to represent vectors. Our initial displacement from point A to point B is represented by the black arrow AB. Note that the arrow has a scaled length that represents the size of the displacement vector, i.e. the distance of 30 km, and the arrow points in the direction of the displacement, i.e. due east. Similarly, our second displacement of 40 km due north is represented by the blue arrow BC.

Our resultant displacement is the net sum of the two displacements. In other words, going from point A to point B, and then from point B to point C, has precisely the same effect, in terms of our resulting position, as if we had gone directly from point A to point C.

This shows us how to add two vectors to determine the resultant vector. We draw the first vector to scale, and point it in the right direction. We draw the second vector to scale, starting with its tail at the head of the first vector, and point the second vector in the direction in which it acts. The resultant vector is then the line between the origin (the tail of the first vector) and the final destination (the head of the second vector).

In vector terminology we would write:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Note that the left side of the above equation is very similar to the left side of the equation for the conservation of linear momentum in that each represents the addition of two vectors.

Displacement (distance in a given direction) is a fairly easy concept to grasp since, as noted above, it's really the basis of map reading. However, it probably isn't obvious what is meant by a momentum vector of mV. So, before we pursue a graphical solution for conservation of momentum, let's first take a look at some other useful properties of vectors.

- Equality - Two vectors are equal if they have the same magnitude and point in the same direction.
- Addition - All vectors involved must have the same units. Vectors may be added by using either a vector triangle or a parallelogram.
- Scalar Multiplication - Multiplication of a vector by a positive constant produces a vector in the same direction but with a different length (different magnitude). Multiplication by a negative constant produces a vector with a different length and pointing in the opposite direction.

We can see that these rules make sense. For example, a vector may be represented by a straight line, drawn to scale to accommodate the vector's magnitude, and pointing in the direction along which the vector acts. It follows that any two lines of the same length, and pointing in the same direction, represent equal vectors.

We can't add apples to oranges, and the same goes for vectors. We can only add vectors of the same kind – displacement vectors, or velocity vectors, or momentum vectors, or…

In Figure 1, we saw how to add two vectors using a vector triangle (AB +BC = AC). We can use the property of vector equality to see how to add vectors using a parallelogram.

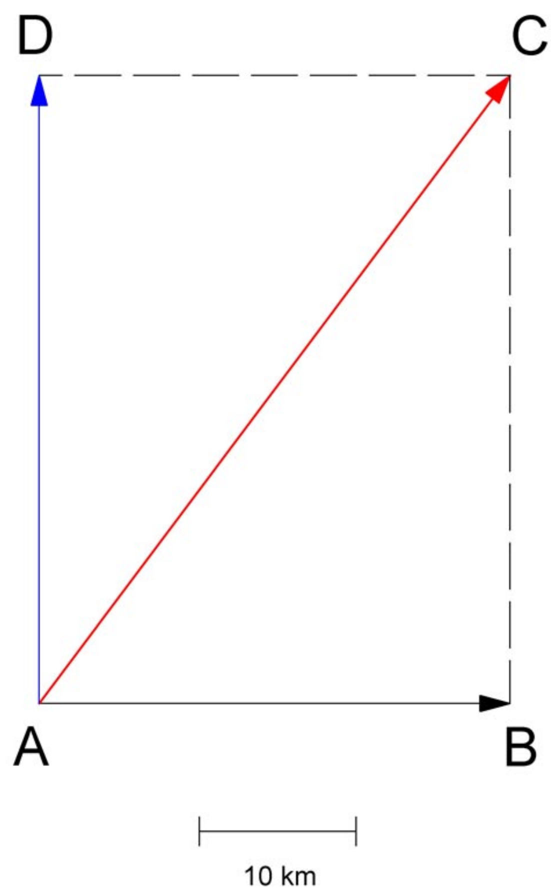


Figure 2 Vector Parallelogram

In Figure 2, note that the vector AD is equal to the vector BC from Figure 1. Both vectors represent displacements of 40 km to the north. We can add vectors AB and AD by constructing a parallelogram, using the two vectors as adjacent sides. The sum of the two vectors (the resultant) is then the diagonal of the parallelogram, AC.

The property of scalar multiplication indicates that, since vector AB represents a distance of 30 km to the east of the origin, then 2 x AB would be a vector twice as long as AB, also pointing to the east. This new vector would, therefore, represent a displacement of 60 km to the east of the origin, which makes sense as being twice (2x) the vector AB.

In the same way, suppose we had a vehicle with a post-impact velocity of 13.9 m/s in an easterly direction. Obviously, we could represent this velocity as a straight line, with its length scaled to 13.9 m/s, and pointing due east. Let's also assume that the mass of the vehicle is 1050 kg. The vehicle's post-impact momentum would be (1050 x 13.9) 14,595 kg m/s in an easterly direction. Note that this is scalar multiplication of a vector quantity (mass x velocity).

Knowing the magnitude of the vehicle's post-impact momentum, and also that the direction of this vector is the same as the vehicle's post-impact velocity, we can represent the momentum vector on a vector diagram as a straight line, with its length scaled to 14,595 kg m/s, and aligned due east.

We now have all the tools necessary to be able to conduct vector analysis, using the pre- and post-impact momenta of two vehicles in a collision. All we have to do is think about how to apply these techniques to solve the momentum equation. Let's do this by means of case study of a real-world crash.

Graphical Analysis

The driver of a 2000 Chevrolet Impala four-door sedan (Vehicle 1) failed to bring the vehicle to a halt at a stop-sign controlled intersection. The front of the Impala struck the left side of a 1998 Saturn SL2 four-door sedan (Vehicle 2) that was northbound along the intersecting road.

Figure 3 2000 Chevrolet Impala

Figure 4 1998 Saturn SL2

After impact, both vehicles travelled across the intersection to the northeast. They ran off the roadway and down a steep embankment. The Saturn rolled over onto its roof. The two vehicles came to rest in close proximity to each other in the ditch.

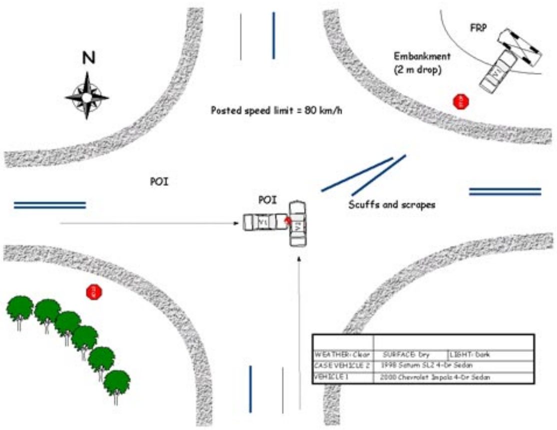


Figure 5 Collision Schematic

As can be seen in the above diagram, both vehicles travelled essentially the same distance after the initial impact. The collision investigators measured the distance travelled from each vehicle's location at the point of impact to its associated final resting position as 25.6 m. They also estimated the effective coefficient of friction to be 0.4.

If we ignore any effects due to the vehicles travelling down the embankment, we can estimate the separation speed of each vehicle by using a simple slide-to-stop calculation based on the measured stopping distance and the friction coefficient.

$$v = 15.9\sqrt{\mu d} \qquad (1)$$

where:

v = separation speed measured in km/h
μ = coefficient of friction
d = stopping distance measured in metres

[Personally, I would use a constant of 4.43 in equation 1 to give the speed in the basic units of m/s. Then I would do all my momentum calculations in the units of kg m/s. But, I know that it drives everyone mad [2] when I do that! So, to keep the peace, I will stick with the units of km/h for speed and kg km/h for momentum.]

In equation 1, the data from our real-world collision are:

$$\mu = 0.4$$
$$d = 25.6 \text{ m}$$

so that:

$$v = 15.9 \sqrt{(0.4 \times 25.6)}$$

$$v = 15.9 \sqrt{10.24} = 15.9 \times 3.2$$

$$v = 51 \text{ km/h}$$

Because both vehicles came to rest after the crash in the same distance, and were subject to the same coefficient of friction, the post-impact speeds of the two vehicles (V_1 and V_2) were the same.

$$V_1 = V_2 = 51 \text{ km/h} \qquad (2)$$

We now have a reasonable estimate of the speed at which each vehicle came away from the collision. What we also need to know is the post-impact velocity of each vehicle. We know the speed, which is the magnitude of the velocity. What we also need, to fully specify velocity, is the post-impact direction of each vehicle.

For the purposes of drawing a vector diagram, we will use the departure angle of each vehicle as it came away from the point of impact.

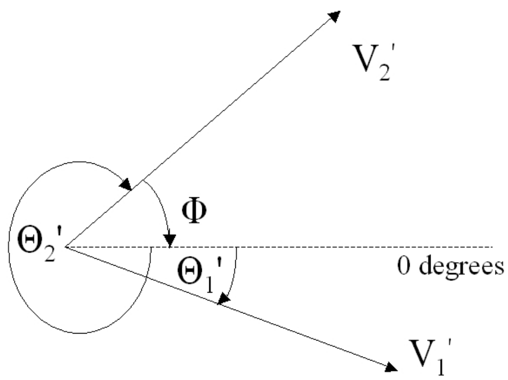


Figure 6 Convention for angular measure

Figure 6 shows the convention adopted by the Society for Automotive Engineers for the measurement of angles. A positive angle is measured clockwise from a datum line.

In the case collision, we will assume that due east is our zero-degree datum line. Based on the physical evidence at the scene (scuff marks and scrapes on the roadway surface) the investigators estimated the departure angle for Vehicle 1 as 329°, and the departure angle for Vehicle 2 as 321°.

[Note that at the collision scene, or using a scaled scene diagram, rather than measuring an obtuse angle such as Θ_2' (Figure 6), the investigators would probably have measured the acute angle Φ and then subtracted this angle from 360° in order to obtain Θ_2'.]

We now have data for the post-impact speed of each vehicle, and its departure angle away from the crash. This provides enough information to calculate the magnitude of the final momentum for each vehicle, and to draw a vector representing this momentum on a vector diagram.

To calculate the magnitude of each final momentum vector, we use scalar multiplication of the vehicle's mass and its post-impact speed.

To draw the vector diagram, we will draw a line to a scale that represents the magnitude of the final momentum, and orient that line on the diagram in the direction of the vehicle's post-impact velocity. This will be the direction of the vehicle's run out from the point of impact as defined by the measured departure angle.

Thus, for the case collision:

Vehicle 1 (Chevrolet Impala)

$m_1 = 1710$ kg
$V_1' = 51$ km/h
$m_1V_1' = 1710 \times 51 = 87{,}210$ kg km/h
$\Theta_1' = 329°$

Vehicle 2 (Saturn SL2)

$m_2 = 1160$ kg
$V_2' = 51$ km/h
$m_2V_2' = 1160 \times 51 = 59{,}160$ kg km/h
$\Theta_2' = 321°$

Figure 7 shows the two vectors m_1V_1' (AB) and m_2V_2' (AD) drawn to a scale of 1 cm being equivalent to 10,000 kg km/h. AB is drawn at 31° above the horizontal axis (i.e. 360 - 329°) to reflect Vehicle 1's departure angle of 329°. Similarly, AD is drawn at 39° above the horizontal axis (i.e. 360 - 321°), this being the departure angle for Vehicle 2.

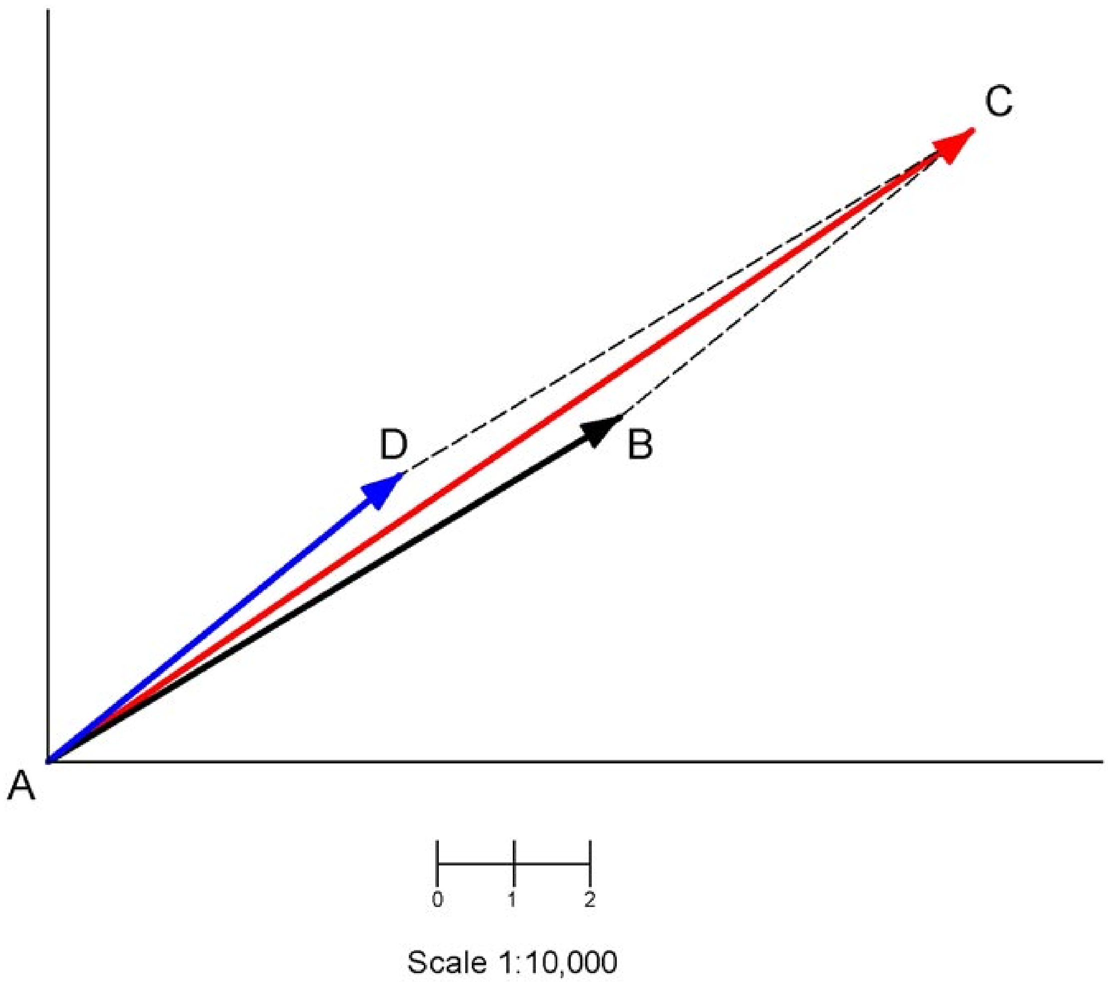


Figure 7 Post-impact momentum vector diagram

While the vector diagram can be drawn by hand on graph paper, or with the use of a drafting table or tablet, it is more convenient to use a computer aided design (CAD) program.

This not only makes the drafting process easier, but also allows direct measurements to be made from the drawing using an inquiry function. This is extremely useful in obtaining the magnitudes of the pre-impact momentum vectors as we will see shortly.

In Figure 7, in addition to representing the two final momentum vectors (AB and AD), we have also constructed the vector parallelogram (ABCD) that allows us to add the final momenta together. The vector AC thus represents the total final momentum of the two vehicles in the collision.

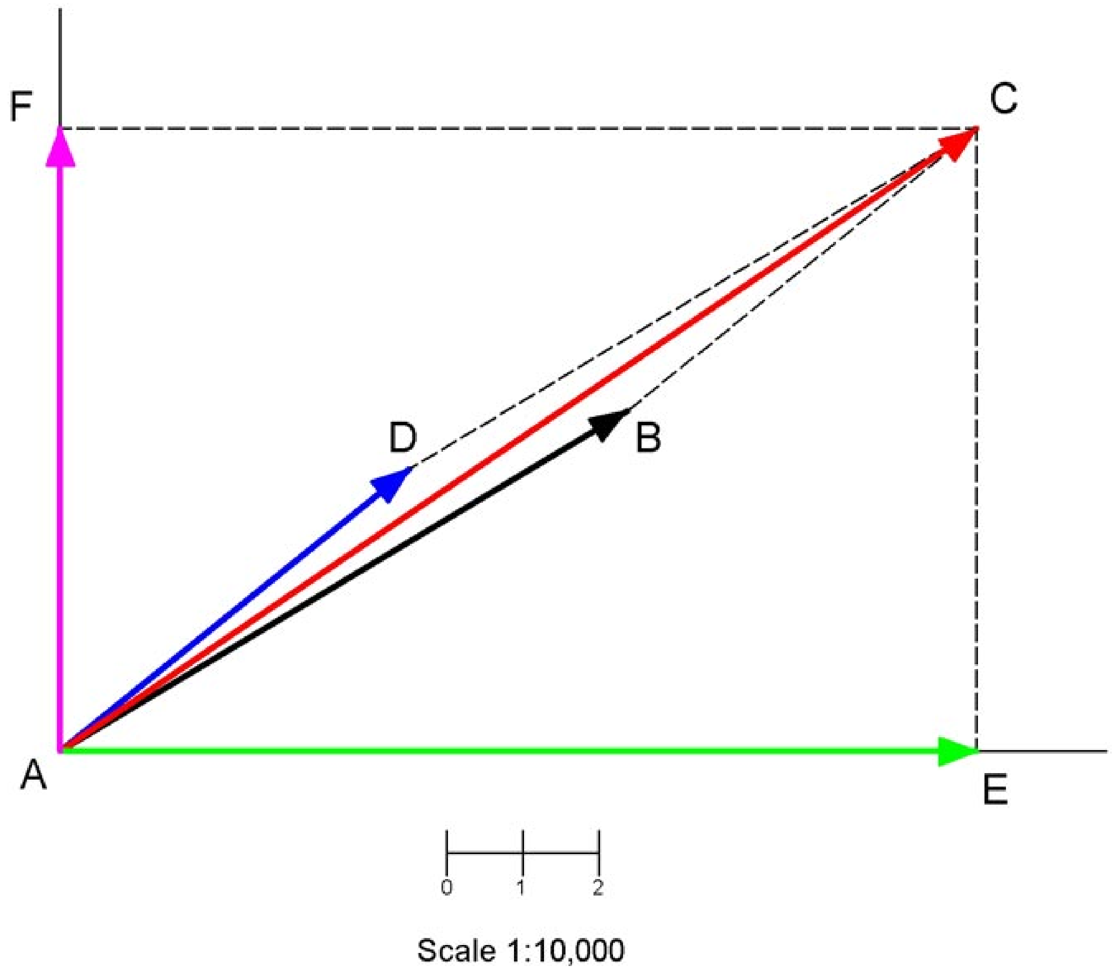


Figure 8 Momentum vector diagram

By the principle of conservation of linear momentum, the total initial momentum of the vehicles must be equal to their total final momentum. It follows that the vector AC must also represent the sum of the two initial momentum vectors.

What we need to do next is determine where the two initial momentum vectors should be located on the diagram.

Referring back to the collision schematic (Figure 5), recall that we chose due east as the zero-degree reference direction for all of our angular measurements. Based on the pre-crash motion of the two vehicles, we know their approach angles to the crash. Vehicle 1, the Chevrolet Impala, was travelling due east and hence had an approach angle of 0º. Similarly, Vehicle 2, the Saturn SL2, was northbound with an approach angle of 270º.

Since the initial velocity of Vehicle 1 must be in the same direction as its pre-impact travel path, it follows that the initial velocity vector for Vehicle 1 must be pointing due east. And, since Vehicle 1's initial momentum is a scalar multiplication of the initial velocity vector, the initial momentum must also be pointing due east. So, on our vector diagram, the vector representing the initial momentum of Vehicle 1 must lie somewhere along the horizontal axis (x-axis), and be pointing due east.

Applying the same logic to Vehicle 2 tells us that the vector representing the initial momentum of Vehicle 2 must lie somewhere along the vertical axis (y-axis) pointing north.

Now we come to the critical step. The initial momentum of Vehicle 1 must be the vector AE, and the initial momentum of Vehicle 2 must be the vector AF. These vectors are both pointing in the appropriate directions, and are the only vectors along these directions that have the appropriate lengths that will add together (using parallelogram AECF) to give the resultant vector AC. Note that the two initial momenta (m_1V_1 and m_2V_2) must add together to give this resultant in order to satisfy the equation of conservation of momentum. Thus, the vector AE represents the initial momentum (m_1V_1) of Vehicle 1, and the vector AF represents the initial momentum (m_2V_2) of Vehicle 2.

Now we use the fact that our vector diagram is drawn to scale and, in particular that 1 cm on the diagram represents a momentum of 10,000 kg km/h.

Using our CAD program, a direct measurement on the diagram shows the length of AE to be 12.0729 cm which thus represents a momentum of 120,729 kg km/h. This is the magnitude of the initial momentum of Vehicle 1, that is the product m_1V_1. Consequently, we have:

$$m_1V_1 = 120{,}729 \text{ kg km/h} \qquad (3)$$

But, we know the mass (m_1) of Vehicle 1 to be 1710 kg. Consequently, equation 3 becomes:

$$1710 \times V_1 = 120{,}729 \text{ kg km/h}$$

so that

$$V_1 = \frac{120{,}729}{1710} = 71 \text{ km/h} \qquad (4)$$

A similar measurement of the length of AF shows the initial momentum (m_2V_2) of Vehicle 2 to be 82,147 kg km/h. Thus:

$$m_2V_2 = 82{,}147 \text{ kg km/h}$$

where $m_2 = 1160$ kg

so that $$V_2 = \frac{82{,}147}{1160} = 71 \text{ km/h} \qquad (5)$$

Thus, equations 4 and 5 give the two initial speeds of Vehicle 1 and Vehicle 2 as:

$$V_1 = 71 \text{ km/h}$$

$$V_2 = 71 \text{ km/h}$$

We have solved the equation of conservation of momentum !

Verification

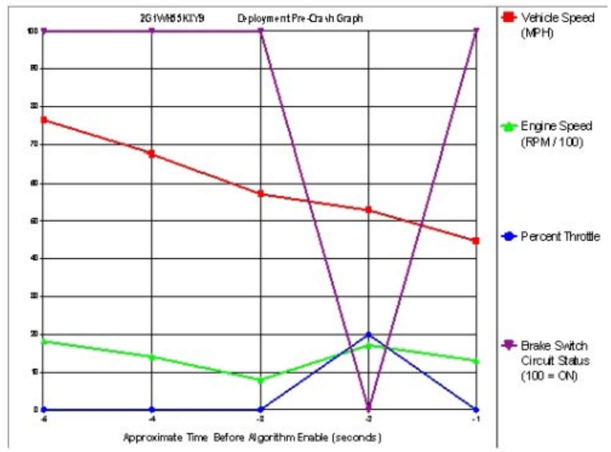
The speeds of both vehicles are somewhat below the speed limit of 80 km/h for the subject roadway. However, note that these speeds are those immediately prior to the crash, i.e. at the point where the vehicles first made contact, Remember that momentum considers the changes in velocity in the crash phase; it looks at the velocities of the vehicles just before and just after the impact. Our calculated speeds do not include the effect of pre-impact braking should this have occurred for one or other of the vehicles. (In such a case, we would need to account for this by means of separate calculations.) Consequently, the travel speeds of the vehicles some seconds before impact could be different from our calculated values.

One thing that is certain is that the Impala did not come to a stop at the intersection. It would not be reasonable for the vehicle to accelerate from a stop to the impact speed in the available distance. Consequently, our analysis supports the thesis that the driver of the Impala ran the stop sign.

In this particular case, this scenario can be confirmed since the Impala is equipped with an event data recorder (EDR) that provides pre-crash data. The EDR reports that one second before algorithm enable (AE), the speed of the Impala was 72 km/h (45 mph). This corresponds very well to our calculated pre-impact speed of 71 km/h.

However, we must realize that, because of the way in which data are stored in the EDR's buffer, AE could have occurred almost immediately after the point at t=-1s was captured, or up to one second later (t=0s). [3]

The agreement between the calculated and recorded values suggests that either the former was the case, or that the vehicle did not slow down appreciably (no or only light braking) up to the point when AE did occur.



Seconds Before AE	Vehicle Speed (MPH)	Engine Speed (RPM)	Percent Throttle	Brake Switch Circuit Status
-5	76	1792	0	ON
-4	68	1408	0	ON
-3	57	832	0	ON
-2	53	1664	20	OFF
-1	45	1344	0	ON

Figure 9 Pre-crash data for Vehicle 1

Note also that that five seconds before AE, the vehicle's speed was 122 km/h (76 mph)! Over the five seconds before the crash the driver was braking and the vehicle's speed was going down (initially at 0.3-0.5 g). Around three seconds to AE the driver released the brake, briefly applied the accelerator (at least 20% throttle), and then went back to the brakes. At no point did the vehicle's speed go down to zero! The driver definitely did not stop for the traffic sign.

One further item to note with respect to the pre-impact speed recorded by the EDR. It is very close to the speed obtained through our momentum analysis. This should do two things. Firstly, it should give us confidence that our graphical solution gives the correct answer. Secondly, we can think of the crash as a real-world experiment in physics. The fact that our calculated result agrees with experimental measurement (the EDR) verifies that the principle of conservation of linear momentum is correct. It also verifies that Newton's laws of motion are valid. Since momentum gives the right answer, and our derivation of momentum was based on Newton's laws, then these laws must also be correct.

However, we should be aware that the case collision is just a single "experiment" for the physics of momentum. In addition, we have made a number of assumptions in computing the run-out speeds of the two vehicles (e.g. the estimate of the coefficient of friction, the vehicle braking efficiencies, the treatment of the travel down the embankment). There is also some uncertainty in the speed of the Impala at impact as determined from the EDR. So, the "good" agreement between the speeds calculated from momentum, and those recorded by the EDR, should be tempered by the knowledge of the potential limitations in these data.

Nevertheless, the case does give us one point of reference. As we continue to conduct more investigations and undertake similar reconstructions, further confirmation of the appropriateness of the technique will be achieved - which will constitute proof of the method - and hence of the physical laws.

Conclusion

So, we accomplished the seemingly impossible. We solved one equation in which there were two unknowns. But, note that this was only possible because we knew both the approach and departure angles of the two vehicles and hence were able to plot the vehicles' momenta in two dimensions.

In particular, once we had drawn the post-impact momentum vectors and determined the total momentum of the two vehicles after the crash, we needed the approach angles of both vehicles to determine the directions of the pre-impact momentum vectors. This allowed us to construct the vector parallelogram (AECF) that would add the two initial momentum vectors to determine the total initial momentum of the vehicles. Knowing the directions in which vectors AE and AF must point allows only one such parallelogram to be drawn. Hence we obtained a unique solution to the momentum equation.

The graphical method of solving the momentum equation is very useful. It's relatively easy to develop the drawing and to take the required measurements (especially if you have a CAD program handy). And the calculations required are very simple, requiring only straightforward mathematics.

The other advantage is that the directions of the momentum vectors reflect the physical situation of the crash. They are pointing in the directions along which the vehicles actually travelled, and so the arrows on the vector diagram look just like the vehicle paths on the collision schematic. This provides a double check on how the final momentum vectors are being added together to determine the resultant, and how the resultant is broken apart to give the initial momentum vectors.

If we adopt a purely algebraic solution to the problem, using mathematics and trigonometrical functions (sines and cosines), the process is not so self-evident.

However, you will get to see if this is indeed the case, should you continue onto the next page from a physicist's notebook that will discuss the algebraic solution for momentum and take a look at some computer-based methods.

References

1. Page from a Physicist's Note-book – Momentum 101 – The principle of conservation of linear momentum; German A, EOTIS; December 12, 2006

2. Page From a Physicist's Notebook: Work is mad - it takes energy!; German A; Proc. CMRSC-XV; Fredericton, New Brunswick; June 5-8, 2005

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