Page From a Physicist's Notebook

Momentum 101

The principle of conservation of linear momentum

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For police officers, the principle of the conservation of linear momentum is the essence of Level IV training for collision reconstructionists. While courses on this topic are often focused on the individual parameters in the momentum equation, and how to use the equation to calculate vehicle speeds, the actual science behind the process is sometimes not well understood.

In this page from a physicist's notebook we will explore familiar equations and concepts (see box at right), and combine these with the appropriate mathematics, to develop the basic equation governing the conservation of linear momentum.

Newton's laws of motion

When two vehicles collide, the resulting dynamics are governed by three well-known physical laws - Newton's Laws of Motion. While everyone can quote these laws more or less correctly, applying them to the physical situation of a motor-vehicle crash, is not always done quite so readily. However, this shouldn't be the case if we think about the collision in basic terms, and apply each law carefully. In fact, if we are to understand momentum, this is precisely what is required. So, here goes…

From basic principles…

Equations of uniform motion
$v = v_0 + at$

Newton's 1st law of motion
An object will remain at rest, or in a state of uniform motion in a straight line, unless acted on by an unbalanced external force.

Newton's 2nd law of motion
$F = ma$

Newton's 3rd law of motion
Action and reaction are equal and opposite
$F_1 = - F_2$

The crash we are going to study is perfectly general in nature. Two vehicles, Vehicle 1 and Vehicle 2, are approaching each other, speeding along a collision course (Figure 1). Neither driver takes any avoiding action. The vehicles run into each other at the point of impact (POI). There is a lot of scraping of metal-on-metal, crushing of the vehicles' structural elements, and breaking glass, all

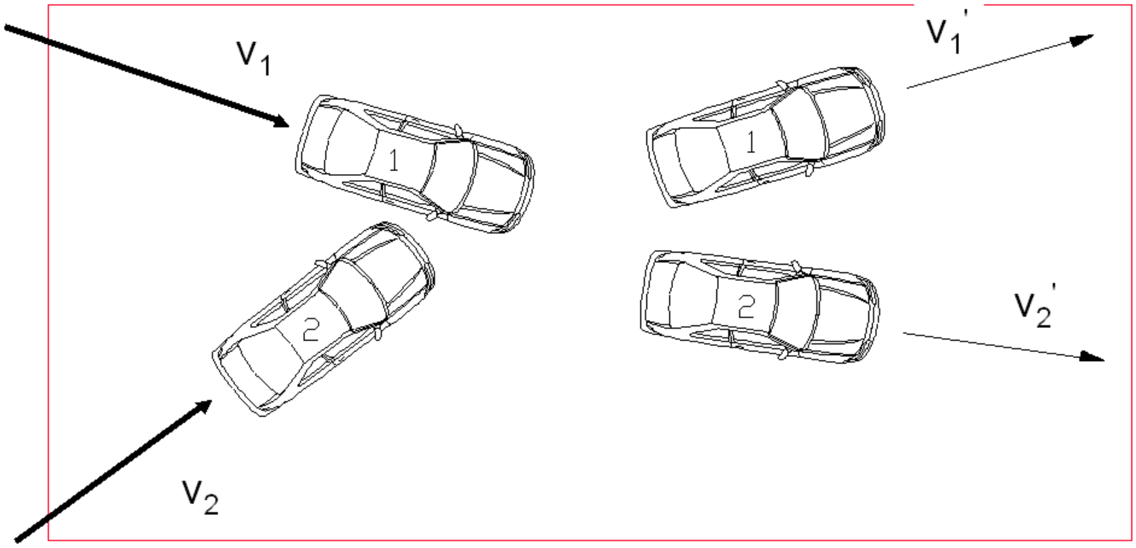

Figure 1. Vehicle-to-vehicle collision (pre- and post-impact conditions)

of which is accompanied by lots of noise (not the least of which results from cursing by one or more of the drivers!) The damage to the vehicles progresses as they engage each other. Each vehicle tries to force its intentions on the other with respect to structural deformation and subsequent motion, with the other trying to resist and apply its own will on the situation. Then, as the point of maximum engagement is reached, the vehicles begin to rebound from the impact, and eventually separate from each other. Each then departs from the point of impact and follows its own path, skidding or rolling down the road, before coming to its final resting position (FRP) in a cloud of steam.

OK, so that's a rather dramatic rendition of a motor vehicle crash, but the critical elements that we need to take into account have all been included. Each vehicle has an initial velocity, i.e. a given travel speed along a direction specified by its approach angle to the collision. Each vehicle experiences a collision force due to its interaction with the other vehicle. And, each vehicle comes away from the collision with a new velocity, i.e. a separation speed in a direction that is specified by a departure angle.

In order to analyze the crash, let's first focus on the situation for Vehicle 1. In particular, let's isolate this vehicle from the collision schematic in Figure 1, and consider its specific situation in the pre-crash and post-crash phases of the collision.

The specific parameters that apply to Vehicle 1 are shown in Figure 2. The vehicle approaches the POI travelling at a speed V_1, in a direction with an approach angle somewhat below the horizontal. After striking Vehicle 2, it leaves the collision, travelling at speed V_1', and with a departure angle somewhat above the horizontal.

Note that we are being very careful to specify both the vehicle's speed and its direction for both the pre- and post-impact conditions. There are two reasons for this. Firstly, the speed and direction of the vehicle are the two parameters that define the vehicle's velocity. Secondly, both of these parameters are likely to have been changed as a result of the crash.

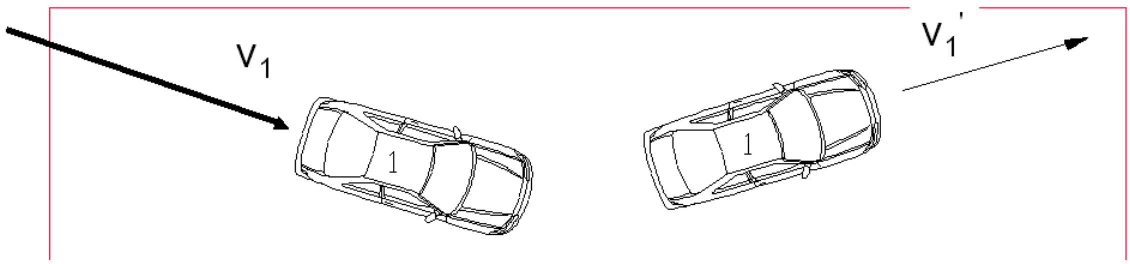

Figure 2. Collision situation for Vehicle 1

It is quite probable that the separation speed of Vehicle 1 is less than its initial speed. The vehicle may well have been slowed down by the collision force as it interacted with Vehicle 2 (i.e. its forward progress was impeded because Vehicle 2 was in the way!)

[However, note that in other collision circumstances, a vehicle may be accelerated, and hence have its speed increased, for example in the case of a stationary vehicle being rear-ended.]

In addition to a probable reduction in speed in the collision, it is certain that the direction of travel of Vehicle 1 was changed. This is readily observed in Figure 2 where the approach and departure angles to and from the POI are quite different.

So, in our general collision situation, both the speed of Vehicle 1, and its direction of travel, have been changed.

The vehicle's initial speed and its pre-impact travel direction define the vehicle's initial velocity (V_1). Similarly, the vehicle's final speed, as it leaves the collision, and its post-impact travel direction, define the vehicle's final (or separation) velocity (V_1').

We have observed that both the speed and travel direction of Vehicle 1 changed as a result of the collision. By definition, this means that the velocity of Vehicle 1 changed during the crash.

In particular, these changes in Vehicle 1's velocity occurred during the time when the two vehicles were in contact. The vehicle underwent a change in velocity over the time of the collision. Thus, the vehicle was accelerated (or decelerated) because of the collision.

One of the equations of uniform motion defines acceleration:

$$v = v_0 + at \qquad (1)$$

where:

v = final velocity
v_0 = initial velocity
a = acceleration
t = time of collision

Thus:

$$a = \frac{v - v_0}{t} \qquad (2)$$

Equation 2 is an expression of the basic definition of acceleration as the rate of change of the vehicle's velocity with time.

We can apply equation 2 to the specific situation for Vehicle 1 in our subject collision to give an expression for that vehicle's acceleration in the crash:

$$a_1 = \frac{V_1' - V_1}{t} \qquad (3)$$

where:

a_1 = acceleration of Vehicle 1 in the collision
V_1' = final (separation) velocity
V_1 = initial velocity
t = time of collision

Now, Newton's first law of motion tells us that a vehicle moving at constant velocity (i.e. travelling at constant speed in a straight line) will continue to do so unless acted on by an unbalanced external force. Clearly, because of the collision, Vehicle 1 did not keep moving at constant velocity. Remember that both its speed and direction of travel were changed as a result of the collision. Thus, the vehicle's velocity did change and, as a consequence, we know that it must have experienced an external force in the crash to cause this. Clearly, this was the force (F_1) exerted on Vehicle 1 by Vehicle 2 while the two vehicles were in contact.

The first law says that a force must have acted on the vehicle

Newton's second law of motion lets us quantify this force. It says that the force seen by Vehicle 1 is the product of the vehicle's mass (m_1) and the acceleration (a_1) that resulted in the crash.

So, we can express Newton's second law (F=ma) for Vehicle 1 in the collision as:

$$F_1 = m_1 a_1 \qquad (4)$$

The second law tells us how big a force acted on the vehicle

And, since, from equation 3, the acceleration (a_1) can be written in terms of the vehicle's final velocity (V_1'), its initial velocity (V_1), and the time taken (t) to change between these velocities, we have:

$$F_1 = m_1 a_1 = m_1 \left(\frac{V_1' - V_1}{t}\right) \qquad (5)$$

$$F_1 = m_1 \left(\frac{V_1' - V_1}{t}\right) \qquad (6)$$

A precisely similar situation applies to Vehicle 2 in the collision (see Figure 3). Note that the final velocity (V_2'), and the initial velocity (V_2), for Vehicle 2 are different, since this vehicle's speed and its direction of travel were also changed as a result of the collision. Because of this change in velocity over the time (t) of the collision, Vehicle 2 must have undergone an acceleration (a_2).

Vehicle 2's acceleration will be given by equation 2 as:

$$a_2 = \frac{V_2' - V_2}{t} \qquad (7)$$

where:

a_2 = acceleration of Vehicle 2 in the collision
V_2' = final (separation) velocity
V_2 = initial velocity
t = time of collision

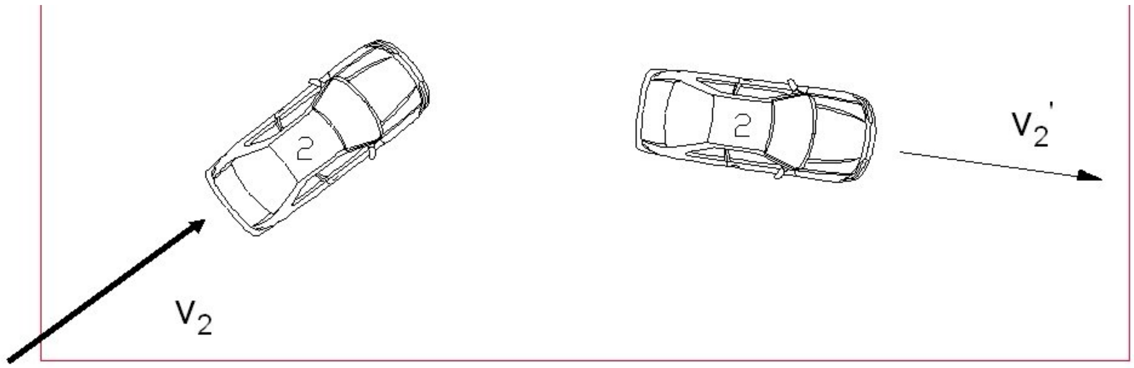

Figure 3. Collision situation for Vehicle 2

And, we know that to produce this acceleration requires that Vehicle 2 experienced a collision force (F_2) given by Newton's second law (F=ma) as:

$$F_2 = m_2 a_2 \qquad (8)$$

As before, we substitute for Vehicle 2's acceleration (a_2) from equation 7 into equation 8 to give the force acting on Vehicle 2 (F_2) as:

$$F_2 = m_2 a_2 = m_2 \frac{(V_2' - V_2)}{t}$$

$$F_2 = m_2 \frac{(V_2' - V_2)}{t} \qquad (9)$$

So, in the collision, forces acted on both vehicles that resulted in changes in their velocities. F_1 was the force that Vehicle 2 applied on Vehicle 1. This force caused the change in velocity of Vehicle 1 in the crash. Similarly, F_2 was the force that Vehicle 1 applied on Vehicle 2 and caused its velocity to change.

We can see that we have two different forces, F_1 and F_2, acting mutually on two different objects, Vehicle 1 and Vehicle 2. In such a case, Newton's third law of motion will apply. Action, which we can take to be the force applied on Vehicle 1 by Vehicle 2, must be equal and opposite to the reaction force, which is then the force applied on Vehicle 2 by Vehicle 1.

Newton's third law:
Action and reaction are equal and opposite

So, to apply the third law to our collision situation, the action force (F_1) has to be equal (in magnitude) and opposite (in direction) to the reaction force (F_2). Mathematically we write this as:

$$F_1 = - F_2 \qquad (10)$$

The equals sign indicates that the two forces have the same size (magnitude), while the negative sign denotes that (since the two forces are vectors) F_2 is acting in the opposite direction to F_1.

In equations 6 and 9 above, we have derived expressions for these two forces in terms of the associated changes in velocity over the time of the collision. If we substitute these two equations into equation 10, we will have:

$$m_1 \frac{(V_1' - V_1)}{t} = - m_2 \frac{(V_2' - V_2)}{t} \qquad (11)$$

Since the duration of the collision, i.e. the time (t) for which the two vehicles are in mutual contact, appears in the denominator of both sides of equation 11, it can be eliminated to give:

$$m_1 (V_1' - V_1) = - m_2 (V_2' - V_2) \qquad (12)$$

Multiplying the terms inside the brackets on both sides of equation 11 then gives:

$$m_1V_1' - m_1V_1 = - m_2V_2' + m_2V_2 \qquad (13)$$

[Note that the product of the last two terms in equation 12 is $(-m_2) \times (-V_2)$. The two negative signs multiply together to create a positive value, i.e. the product is $+ m_2V_2$.]

If we add m_1V_1 to both sides of equation 13, we get:

$$m_1V_1'-m_1V_1+m_1V_1 = - m_2V_2'+m_2V_2+m_1V_1$$

so that

$$m_1V_1' = - m_2V_2' + m_2V_2 + m_1V_1 \qquad (14)$$

Similarly, adding m_2V_2' to both sides of equation 14 gives:

$$m_1V_1' + m_2V_2' = - m_2V_2' + m_2V_2 + m_1V_1 + m_2V_2'$$

so that:

$$m_1V_1' + m_2V_2' = m_2V_2 + m_1V_1$$

Re-arranging the terms slightly so that the parameters for Vehicle 1 come first on both sides of the equation gives:

$$m_1V_1' + m_2V_2' = m_1V_1 + m_2V_2$$

and finally, swapping the two sides of the equation, moves the initial parameters (the pre-impact vehicle velocities) to the left and the final values (the post-impact vehicle velocities) to the right:

$$m_1V_1 + m_2V_2 = m_1V_1' + m_2V_2' \qquad (15)$$

Great! But, what's the significance of m_1V_1, m_2V_2, etc. ?

Momentum

Equation 15 is one of those things that physicists strive for – a "beautiful" (i.e. simple) formulation of the laws of nature – in this case a law governing the interaction of two vehicles involved in a collision.

To understand its significance, we need to think about how Newton actually expressed his second law of motion.

We generally think of it in a simple formulation, F = ma (force is equal to mass times acceleration).

Newton was aware that the mass of an object and its velocity are important parameters in describing the object's motion and, in particular, changes to its motion. It is difficult to start a large object moving; it's hard to stop something once it is moving. It's really hard to get a large object to move quickly, and very difficult to bring a fast-moving, large object to rest.

Newton's original treatise, "*Philosophiae Naturalis Principia Mathematica*", written in 1687, stated (with translation from the original Latin [1]):

> *Lex II: Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.*
>
> Law II: The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed. — If a force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

Newton talked about force producing a proportional change in the object's motion. In modern terms, we would paraphrase his law as:

> The rate of change of momentum of a body is proportional to the resultant force acting on the body and is in the same direction.

[If we use SI units - Newtons, kilograms, and metres per second squared - the constant of proportionality is unity. Force is then equal to the rate of change of momentum. Hence the formulation of the second law as: F = ma.]

Momentum is the combination of mass and velocity. It is actually the product of these two parameters. Applying a force will cause a change in momentum, and the rate of this change will be directly related to the degree of force applied.

We might note that this is precisely what equation 6 tells us about the motion of Vehicle 1 as a result of the crash:

$$F_1 = m_1 \frac{(V_1' - V_1)}{t} \qquad (6)$$

which we can expand as:

$$F_1 = \frac{(m_1V_1' - m_1V_1)}{t} \qquad (6)$$

If momentum is the product of mass and velocity, then m_1V_1' is the momentum of Vehicle 1 after impact, that is this vehicle's final momentum. Similarly, m_1V_1 is the momentum of the vehicle before the collision, or its initial momentum.

[1] http://en.wikipedia.org/wiki/Newton's_laws_of_motion

$m_1V_1' - m_1V_1$ is the change in momentum (the final momentum minus the initial momentum) that occurred in the collision.

$(m_1V_1' - m_1V_1)/t$ is the rate of change of momentum with time, i.e. the change in momentum divided by the time taken to make the change.

Since $F_1 = (m_1V_1' - m_1V_1)/t$, the magnitude of the force is given by the rate of change of the vehicle's momentum.

Note also, Newton's statement that the change in motion "is made in the direction of the right line in which that force is impressed". This means that the change in momentum, and the vehicle's acceleration, will take place along the line of action of the applied force (i.e. that of the principal direction of force or PDOF).

In vector notation, we would write:

$$\vec{F_1} = m_1 \vec{a_1} = m_1 \frac{(\vec{V_1}' - \vec{V_1})}{t}$$

acceleration, and velocity are all vector quantities - with both magnitudes - and direction.

Conservation of Momentum

We have seen that momentum is defined as the product of the mass and velocity of an object.

In the case of a motor vehicle collision the mass of a vehicle generally remains the same (unless it splits in two – in which case we must treat the two post-crash halves separately!) However, the pre-impact and post-impact speeds and heading angles of any given vehicle often differ dramatically.

We generally think of vehicles as being slowed down abruptly when they come into collision, but as we noted earlier, under certain conditions one of the partner vehicles may well be accelerated and thus gain speed as a result of a crash.

Just before an impact takes place, the initial momentum of a vehicle will be the product of its mass and its initial velocity (mV). Immediately after impact, the vehicle will have reached some final (or separation) velocity, and its final momentum will be the product of the mass and the final velocity (mV').

Earlier, we used Newton's laws of motion and one of the equations of uniform motion to derive equation 15:

$$m_1V_1 + m_2V_2 = m_1V_1' + m_2V_2' \quad (15)$$

We should note that the first two terms (m_1V_1 and m_2V_2) are the initial momenta of Vehicle 1 and Vehicle 2. Consequently, their sum, on the left side of the equation, represents the total momentum of the two vehicles prior to the crash.

Similarly, the two terms of the right side of the equation ($m_1V_1' + m_2V_2'$) are the final momenta of Vehicle 1 and Vehicle 2. The sum of these, on the right side of the equation, represents the total momentum of the two vehicles after the crash.

Equation 15 tells us that the total momentum of the vehicles before the crash is equal to their total momentum immediately after the crash. No momentum is lost. Momentum is conserved.

Thus equation 15 is a statement of the Principle of Conservation of Linear Momentum.

$$m_1V_1 + m_2V_2 = m_1V_1' + m_2V_2'$$

The Principle of Conservation of Linear Momentum

Now, before we rush off to put numbers into this equation, we need to recall that momentum is a vector quantity. It's the product of a scalar (vehicle mass) and a vector (the vehicle's velocity). Because of the latter, the resulting momentum is also a vector quantity.

We know that vectors have two important properties – magnitude (size) – and direction. Consequently, we can't just deal with the scalar quantities (the vehicle masses) and the sizes of the vector quantities (the vehicle speeds). When we have vector quantities we must use vector analysis – a process that takes into account both the magnitudes and the directions of the involved vectors.

Vector analysis is a little bit more complicated than regular mathematics but we can approach it step-by-step and it will be easy to see how to apply the method to a motor-vehicle collision. However, after struggling through nine pages of scientific explanations and mathematical derivations, it's time for a break. So, we will leave the topic of vector analysis for another page from a physicist's notebook!

Author

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