Page From a Physicist's Notebook

Falls and Vaults – These really are rocket science!

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Normally, the catch phrase for the physics and mathematics that are necessary to perform calculations related to motor vehicle collisions is that "It's not rocket science!" However, the subject of falls and vaults is – more-or-less – rocket science. They both involve consideration of a projectile (a vehicle, an occupant – or a motorcycle rider) being launched from the Earth's surface into the air and following a free-flight trajectory under the action of gravity. We generally want to calculate the speed at take-off based on the angle of projection, and the horizontal and vertical distances travelled between the points of take-off and landing. This is basically the opposite of the computations made by rocket scientists, or artillery commanders, who know the "muzzle velocity" of the rocket or shell and need to calculate the angle of projection in order for the projectile to land on a target with a known (or measured) range.

As usual, in our *Page from a Physicist's Notebook,* we will use physics and mathematics obtained from first principles to derive the equations that can be applied directly to motor vehicle crashes involving falls and vaults. And, in doing so, we will find that rocket science is basically a piece of cake!

The information that we need (and that we already know!) is shown in the side bar at the right of the page. We will use one of the equations of uniform motion to develop equations for the horizontal distance travelled (the run), and for the vertical distance travelled (the rise) in terms of the time of flight. Newton's second law will allow us to determine the components of the object's acceleration along the horizontal and vertical axes. And, because the object

From basic principles…

Equations of uniform motion

$$d = v_0 t + \tfrac{1}{2}at^2$$

$$2ad = v^2 - v_0^2$$

Newton's laws of motion

$$F = ma$$

$$W = mg$$

Acceleration due to gravity

$$g = 9.81 \text{ m/s}^2$$

Trigonometrical functions:

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\sin\theta}{\cos\theta}$$

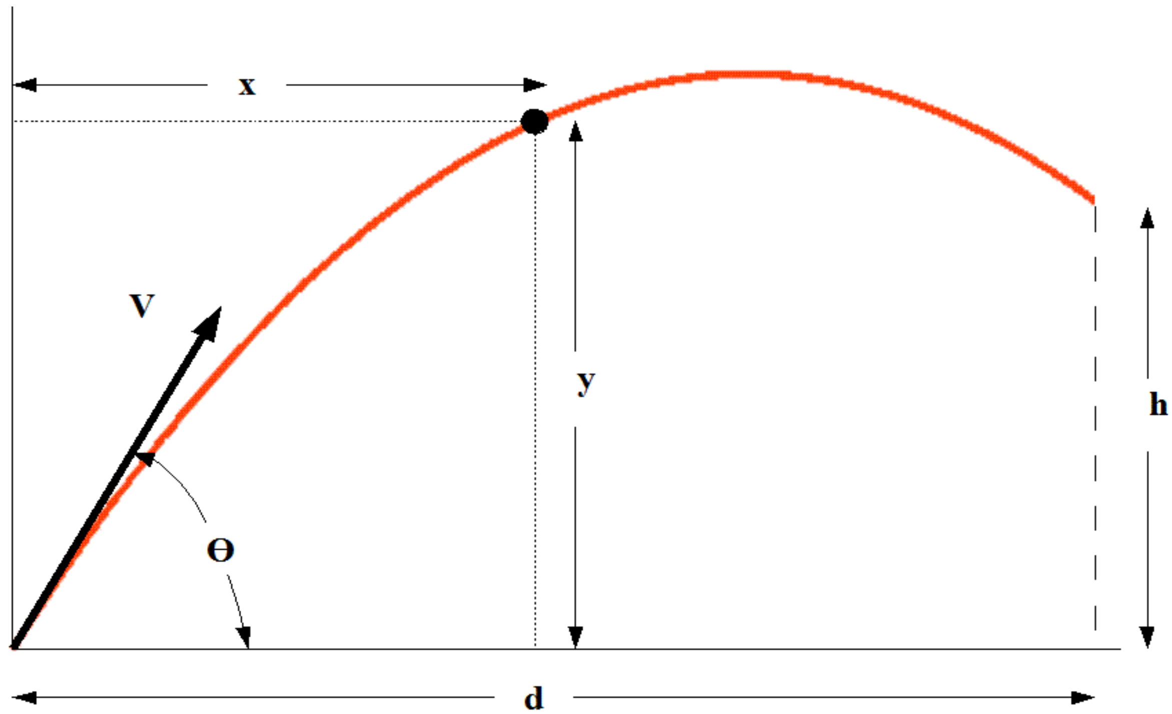

Figure 1. Path of projected object

may be projected at any angle, we will find another use for the well-known trigonometrical relationships for sine, cosine and tangent.

Trajectory Analysis

The general case of an object being projected, and travelling in an arc under the action of gravity, is shown in Figure 1. The initial conditions, and the pertinent parameters of the trajectory, are as follows:

V = initial speed at the point of take-off
Θ = angle of projection with respect to the horizontal (x-axis)
d = horizontal distance (run) between take-off and landing
h = vertical distance (rise) between take-off and landing

The equations for uniform motion provide the relationships between time, distance, speed, and acceleration. Consequently, we can use these equations as they apply to the object's trajectory as a projectile.

However, since the object is moving along a two-dimensional path, we can consider the motion with respect to the x- and y-axes separately. But, in order to do so, we first need to establish the x- and y-components of the object's initial velocity.

Figure 2 shows the horizontal component of the initial velocity as AB, and the vertical component as AD. From the right-angled triangle ABC, we can formulate the sine and cosine of the angle of projection, Θ, as:

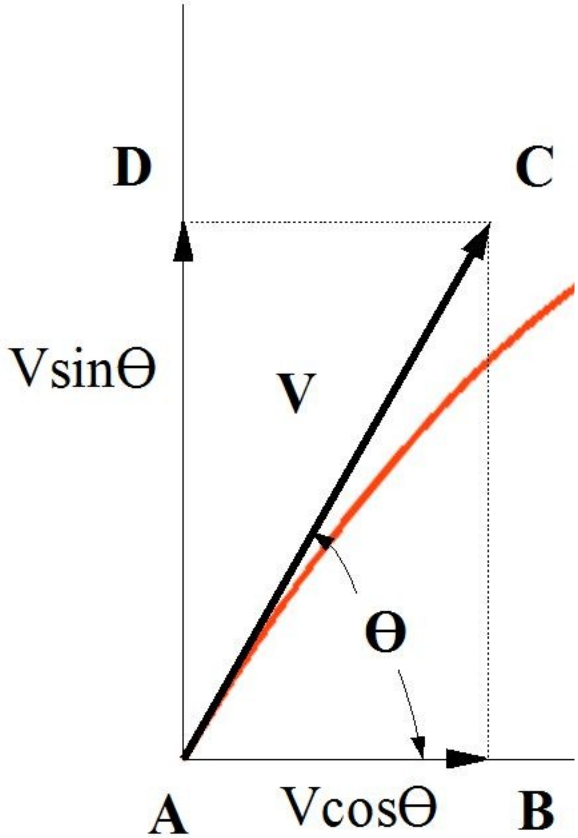

Figure 2. X- and Y-Components of the initial velocity

For those unfamiliar with moving terms from one side to the other, or cross-multiplying an equation, the specific step taken at each stage will be placed inside square brackets. Thus [x V] will indicate that we are to multiply each side of the equation by V.

Clearly, if the left side of the equation starts off as cos Θ, the result of multiplying this term by V will be:

$$\cos\Theta \times V = \cos\Theta\ V = V\cos\Theta$$

Similarly, if the right side of the equation is initially AB/V, multiplying by V will simplify this to:

$$\frac{AB}{V} \times V = AB \times 1 = AB$$

In the following text, we will use square brackets to show the operation being performed, but will leave out any intermediate steps, such as cancelling like terms (e.g. V/V = 1).

$$\cos\Theta = \frac{AB}{AC} = \frac{AB}{V}$$

Re-arranging equations

The basic feature of an equation is that the left side term(s) is equal to the right side term(s). And, the equality will be maintained if we perform the same mathematical operation (e.g. +, -, x or /) to both sides. Consequently, we can use specific operations, applied to both sides of the equation, to re-arrange and/or simplify any given equation.

We can re-arrange the foregoing equation for cos Θ to give an expression for the side of the triangle AB:

[x V]

$$AB = V\cos\Theta \qquad (1)$$

Similarly:

$$\sin\Theta = \frac{BC}{AC} = \frac{AD}{AC} = \frac{AD}{V}$$

[x V]

$$AD = V\sin\Theta \qquad (2)$$

Horizontal Motion

In Figure 1, the black dot represents the position (x,y) of the object along the trajectory at any given time (t).

First, let's consider the motion of the object in the horizontal direction. Newton's second law (F=ma) gives the relationship between the force acting on the object, its mass, and the acceleration that is produced.

In the horizontal direction, the only force acting on the object is air resistance, i.e. the drag produced by the air as the object moves forward horizontally.

Note that there is a force acting on the vehicle - the force due to gravity (weight). But, this force acts vertically downwards, and has no effect on the horizontal motion.

For cars – and humans – flying through the air at the speeds normally involved in motor vehicle crashes, air resistance is small, and can be considered to be zero.

Why is air resistance negligible?

Compare the stopping distance of a vehicle coasting to a halt on a level roadway, with the transmission in neutral, and that of a similar vehicle undergoing locked four-wheel braking. The vehicle that was initially coasting will take a much longer distance to stop because the only drag forces are air resistance and the rolling resistance of the tires and axles. Clearly, these forces are much smaller than those involved in braking. If the vehicle is airborne, there isn't even any rolling resistance!

Now, if the force acting in the horizontal direction is zero, Newton's second law tells us that the acceleration in the horizontal direction (a_x) will necessarily be zero (a_x =F/m =0/m = 0).

So, we can now use one of the equations for uniform motion to relate time, distance, speed and acceleration in the horizontal direction (x-axis):

$$d = v_0 t + \tfrac{1}{2} a t^2$$

where, in the x-direction:

d = x (distance travelled from the origin)

v_0 = V cos Θ (initial horizontal speed)

t = time of flight (to the point x,y)

a = 0 (zero acceleration horizontally)

Consequently:

$$x = V \cos \Theta \, t + \tfrac{1}{2}\, 0 \, t^2$$

$$x = V \cos \Theta \, t$$

[Note that this is the equation for the distance travelled for an object moving at constant speed (d = v t) where, in this case, the speed is the horizontal component of the initial projection velocity. Since there is effectively no horizontal drag force, the object continues moving horizontally at its original (horizontal) speed.]

It follows that the time of flight, t, is given by:

[/ V cos Θ]

$$t = \frac{x}{V \cos \Theta} \qquad (3)$$

Vertical Motion

Now, let's consider how the object moves along the vertical axis.

We have already noted that the gravitational force (the object's weight, W=mg) acts vertically downwards. This will cause the object to accelerate downwards with the acceleration due to gravity ($g = 9.81\ m/s^2$).

As before, we can apply the equation of uniform motion, but this time in the vertical direction (y-axis):

$$d = v_0 t + \frac{1}{2}at^2$$

where, in the y-direction:

$d = y$ (distance travelled from the origin)

$v_0 = V \sin \Theta$ (initial vertical speed)

t = time of flight (to the point x,y)

$a = -g$ (vertical acceleration)

Now, someone is thinking that there must be a mistake here, because we have a negative sign in front of the gravitational acceleration (i.e. $a = -g$).

But, let's go back and look at Figure 1. We have effectively chosen x to be positive moving to the right of the origin, and y to be positive moving vertically upwards. Now, since we know that gravity acts vertically downwards, we need to use the negative sign to show that g is acting in the opposite direction to our positive sign convention.

This negative sign does make sense in terms of how the object moves in the vertical direction. Initially, the object is projected vertically upwards at a speed of V sin Θ. Gravity immediately acts to start reducing the vertical speed. As the projectile moves down range (along the x-axis), it's vertical motion slows, and eventually stops. The projectile has reached its maximum height, the apogee of the arc. The projectile continues to move down range, but now gravity pulls it back and accelerates it towards the earth, i.e. y starts to decrease. At some point, y will become zero and, on a level surface, this will be the landing point. However, if the land drops away from the point of take-off (e.g. there is some sort of depressed ridge or cliff downstream), as x continues to increase, y will now become negative, and the point of landing will be below the point of take-off And, all this because of one little negative sign – or vice versa!

So, using the above-noted values in the equation for uniform motion along the y-axis gives:

$$y = V \sin \Theta \ t + \frac{1}{2}(-g)t^2$$

$$y = V \sin \Theta \ t - \frac{1}{2} g t^2 \qquad (4)$$

Combined Motion

Equations 3 and 4 give the horizontal and vertical distances travelled at any point in time. But, not being witnesses to the motor vehicle crash, and the associated projectile motion, we are not in any position to obtain (measure) any time related to the flight. Consequently, we use the simultaneous Equations 3 and 4 to eliminate t.

From Equation 3:

$$t = \frac{x}{V \cos \Theta}$$

Substitute for t in Equation 4:

$$y = V \sin \Theta \; t - \tfrac{1}{2} g t^2$$

$$y = V \sin \Theta \frac{x}{V \cos \Theta} - \tfrac{1}{2} g \left(\frac{x}{(V \cos \Theta)^2}\right)^2$$

$$y = \frac{x \sin \Theta}{\cos \Theta} - \tfrac{1}{2} g \frac{x^2}{V^2 \cos^2 \Theta} \qquad (5)$$

Note that by eliminating time, we have created an expression for the vertical distance travelled, y, in terms of the horizontal distance travelled, x. This equation describes the path travelled by the projectile. Since y is a quadratic function of x (note the x squared), mathematically this describes a parabola; hence the familiar arced path.

Equation 5 can be used to plot the path taken by the projectile, but for our purposes, we wish to be able to calculate V, the initial speed at the point of take off.

This is simply (!) a matter of rearranging Equation 5:

$$y = \frac{x \sin \Theta}{\cos \Theta} - \tfrac{1}{2} g \frac{x^2}{V^2 \cos^2 \Theta}$$

[$+ \tfrac{1}{2} g \; x^2 / V^2 \cos^2 \Theta$]

$$y + \tfrac{1}{2} g \frac{x^2}{V^2 \cos^2 \Theta} = \frac{x \sin \Theta}{\cos \Theta}$$

[- y]

$$\tfrac{1}{2} g \frac{x^2}{V^2 \cos^2 \Theta} = \frac{x \sin \Theta}{\cos \Theta} - y$$

But $\sin \Theta / \cos \Theta = \tan \Theta$, so that:

$$\tfrac{1}{2} g \frac{x^2}{V^2 \cos^2 \Theta} = x \tan \Theta - y$$

[x 2/g]

$$\frac{x^2}{V^2 \cos^2 \Theta} = \frac{2}{g} \left(x \tan \Theta - y \right)$$

[Reciprocate (invert) both sides]

$$\frac{V^2 \cos^2 \Theta}{x^2} = \frac{g}{2} \frac{1}{(x \tan \Theta - y)}$$

[multiply by $x^2 / \cos^2 \Theta$]

$$V^2 = \frac{x^2 g}{\cos^2 \Theta \; 2 \, (x \tan \Theta - y)}$$

Since we know that g = 9.81 m/s², we have:

$$V^2 = \frac{9.81}{2} \frac{x^2}{\cos^2 \Theta \, (x \tan \Theta - y)}$$

$$V^2 = \frac{4.7 \, x^2}{\cos^2 \Theta \, (x \tan \Theta - y)}$$

Since $\tan \Theta = \sin \Theta / \cos \Theta$, we have:

$$V^2 = \frac{4.7 \, x^2}{\cos^2 \Theta \left(\frac{x \sin \Theta}{\cos \Theta} - y\right)}$$

so that:

$$V^2 = \frac{4.7 \, x^2}{x \sin \Theta \cos \Theta - y \cos^2 \Theta}$$

[Take the square root]

$$V = \frac{2.21 \, x}{\sqrt{(x \sin \Theta \cos \Theta - y \cos^2 \Theta)}}$$

Rather than using the general point x,y along the projectile's path, we may substitute the specific values that correspond to the point of landing.

In this specific case (see Figure 1):

$x = d$ (the run)

$y = h$ (the rise)

giving:

$$V = \frac{2.21\ d}{\sqrt{(d \sin \Theta \cos \Theta - h \cos^2 \Theta)}} \qquad (6)$$

Note that, because d and h are in m, and g is in m/s², Equation 6 gives the initial speed at take-off (V) in m/s. If we wish to have an equation that calculates the initial speed directly in km/h (S), we use the fact that 1 m/s = 3.6 km/h:

$$S = 3.6 \text{ x } V = \frac{3.6 \text{ x } 2.21\ d}{\sqrt{(d \sin \Theta \cos \Theta - h \cos^2 \Theta)}}$$

[Note that, in the above equation "x" is the multiplication symbol, not a distance along the x-axis as defined previously.]

$$S = \frac{7.97\ d}{\sqrt{(d \sin \Theta \cos \Theta - h \cos^2 \Theta)}} \qquad (7)$$

Apogee – Maximum Height

As discussed earlier, as the projectile moves down range, gravity reduces the initial vertical velocity. If the trajectory continues, the projectile's vertical velocity will be reduced to zero and, subsequently, the vertical speed will start to increase as the projectile is pulled back towards the ground.

The maximum vertical height of the trajectory is termed the apogee. We can derive an equation for this value by using another of the equations for uniform motion:

$$2ad = v^2 - v_0^2$$

When, the projectile has reached its maximum height, the final (vertical) velocity, v, is zero. Thus, the parameters in the above equation are:

$a = -g$ (gravitational acceleration, negative in our sign convention)

$d = h_{max}$ (maximum height reached)

$v = 0$ (vertical speed zero at the apogee)

$v_0 = V \sin \Theta$ (initial vertical speed)

Consequently:

$$2(-g)\, h_{max} = 0^2 - (V \sin \Theta)^2$$

$$-2\, g\, h_{max} = -V^2 \sin^2 \Theta$$

[x -1]

$$2\, g\, h_{max} = V^2 \sin^2 \Theta$$

[/ 2g]

$$h_{max} = \frac{V^2 \sin^2 \Theta}{2g} \qquad (8)$$

Note that if the projectile takes off horizontally($\Theta = 0$), the apogee will be zero since sin (0) = 0. Physically, this makes sense since a projectile that is initially moving horizontally can't go up!

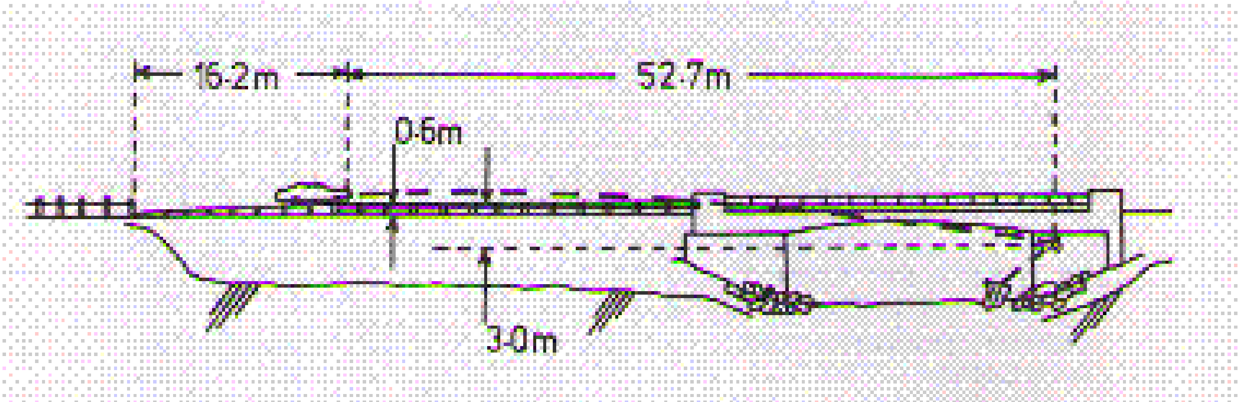

Figure 3. Vault at a bridge guide rail

Case Study of a Vault

A vehicle ran off the right side of the roadway surface and struck the upstream end of a buried steel guide rail attached to a bridge (see Figure 3).

The guide rail rose 0.6 m vertically over a horizontal distance of 16.2 m. The vehicle ramped up the guide rail and became airborne. It subsequently impacted a concrete wing wall on the far side of the bridge, having travelled 52.7 m horizontally, and dropped 3.0 m vertically below the point of take-off.

The angle of projection is effectively the angle of the guide rail.

$$\tan \Theta = \frac{0.6}{16.2} = 0.037$$

Thus, $\Theta = 2.1^{o}$

In Equation 7, we have:

$d = 52.7$ m
$h = -3.0$ m
$\Theta = 2.1^{o}$

so that:

$$\sin \Theta = \sin (2.1^{o}) = 0.037$$

$$\cos \Theta = \cos (2.1^{o}) = 0.999$$

$$S = \frac{7.97\ d}{\sqrt{(d \sin \Theta \cos \Theta - h \cos^2 \Theta)}}$$

$$S = \frac{7.97 \times 52.7}{\sqrt{(52.7 \times 0.037 \times 0.999) - (-3.0)\ 0.999^2}}$$

$$S = \frac{420.02}{\sqrt{(1.93 + 2.996)}}$$

$$S = \frac{420.02}{\sqrt{(4.93)}} = \frac{420.02}{2.22} = 189 \text{ km/h}$$

Falls – A Special Case of Vaults

So far, we have developed a general equation for the motion of a projectile in free flight, the so-called vault. A special case of a vault, called a fall, is when the angle of projection (Θ) is zero, i.e. the object takes off in a horizontal direction (see Figure 4).

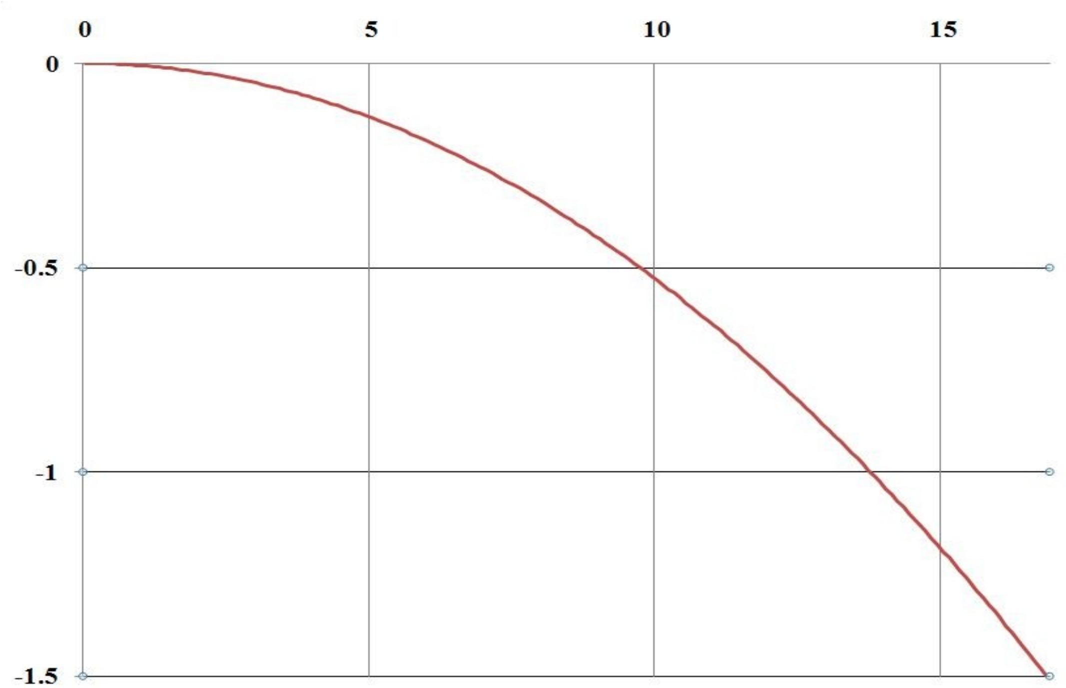

Figure 4. Fall trajectory

Clearly, this situation can only occur when there is some sort of drop off between the point of take-off and the point of landing. The object must land below the point of take-off (i.e. h must be negative according to our sign convention) since, once the object is projected, gravity will immediately pull it downwards.

The fact that, in a fall, $\Theta = 0$ results in simplified form of Equation 7 since $\sin(0) = 0$ and $\cos(0) = 1$:

$$S = \frac{7.97\,d}{\sqrt{(d \sin\Theta \cos\Theta - h \cos^2\Theta)}}$$

For a fall, $\Theta = 0$, so that:

$$S = \frac{7.97\,d}{\sqrt{(d \times 0 \times 0 - h \times 1^2)}}$$

$$S = \frac{7.97\,d}{\sqrt{(-h)}} \text{ km/h} \qquad (9)$$

There's that negative sign again! And this time, it's under a square root sign.

But, remember, it's all because of our sign convention that vertically upwards is positive. So, as discussed above, in a fall, h will necessarily be a negative value. And, if h is negative, -h is positive, so there will not be a problem in taking the square root.

The initial speed (S) will then also be a positive value – exactly as we would expect.

Case Study of a Fall

As shown in Figure 4, a vehicle ran off a horizontal road surface and dropped onto an adjacent roadway allowance that was 1.5 m below grade. The vehicle travelled a horizontal distance of 16.8 m before landing on the roadway allowance.

In this case:

$d = 16.8$ m
$h = -1.5$ m
$\Theta = 0°$

Since this is a fall ($\Theta = 0$), the initial speed at take-off is given by Equation 9:

$$S = \frac{7.97\ d}{\sqrt{(-h)}}$$

$$= \frac{7.97 \times 16.8}{\sqrt{(-(-1.5))}}$$

$$= \frac{7.97 \times 16.8}{\sqrt{(1.5)}}$$

$$= \frac{133.9}{1.22}$$

$S = 109$ km/h

Both of the case studies presented here are events in real-world collisions. This second crash also involved a second horizontal launch of the vehicle, this time from the roadway allowance into a ravine. The parameters for the second trajectory were:

$d = 29.0$ m
$h = -5.2$ m
$\Theta = 0°$

The calculation of the take-off speed in this instance is left as an exercise for the reader.

Computer-Based Solutions

We have derived Equation 7 as the generic expression for speed at take-off in a vault situation, and have shown that the fall equation is just a special case ($\Theta = 0$) of a vault. Consequently, we can use Equation 7 for any fall or vault, so long as we follow the adopted sign convention that h is positive vertically upwards from the point of take-off, and Θ is positive above the horizontal.

It is a simple matter to program Equation 7 into a computer spreadsheet and allow the computer to perform the necessary calculations on a set of parameters entered into the spreadsheet. Such a spreadsheet is shown in Figure 5.

The data entry fields for Θ, d, and h are cells I8, I10 and I12 respectively (and highlighted in blue). To perform a speed calculation, all we need to do is to enter the three relevant values into these cells. The spreadsheet automatically calculates the result, which is shown, rounded to a whole number, in cell I26 (in the final bold-faced line of text).

The value of sin Θ is calculated in cell I17 using the expression =SIN(RADIANS(I8)). Note that the angle of projection, Θ, is first converted from degrees into radians using Excel's RADIANS function since the SIN function has to have its argument expressed in radians.

In a similar manner, the value of cos Θ is calculated in cell I19 using the expression =COS(RADIANS(I8)).

The speed at take-off (V) is calculated in cell I21 using the expression =(7.97*I10)/(SQRT((I10*I17*I19)-(I12*I19*I19)))

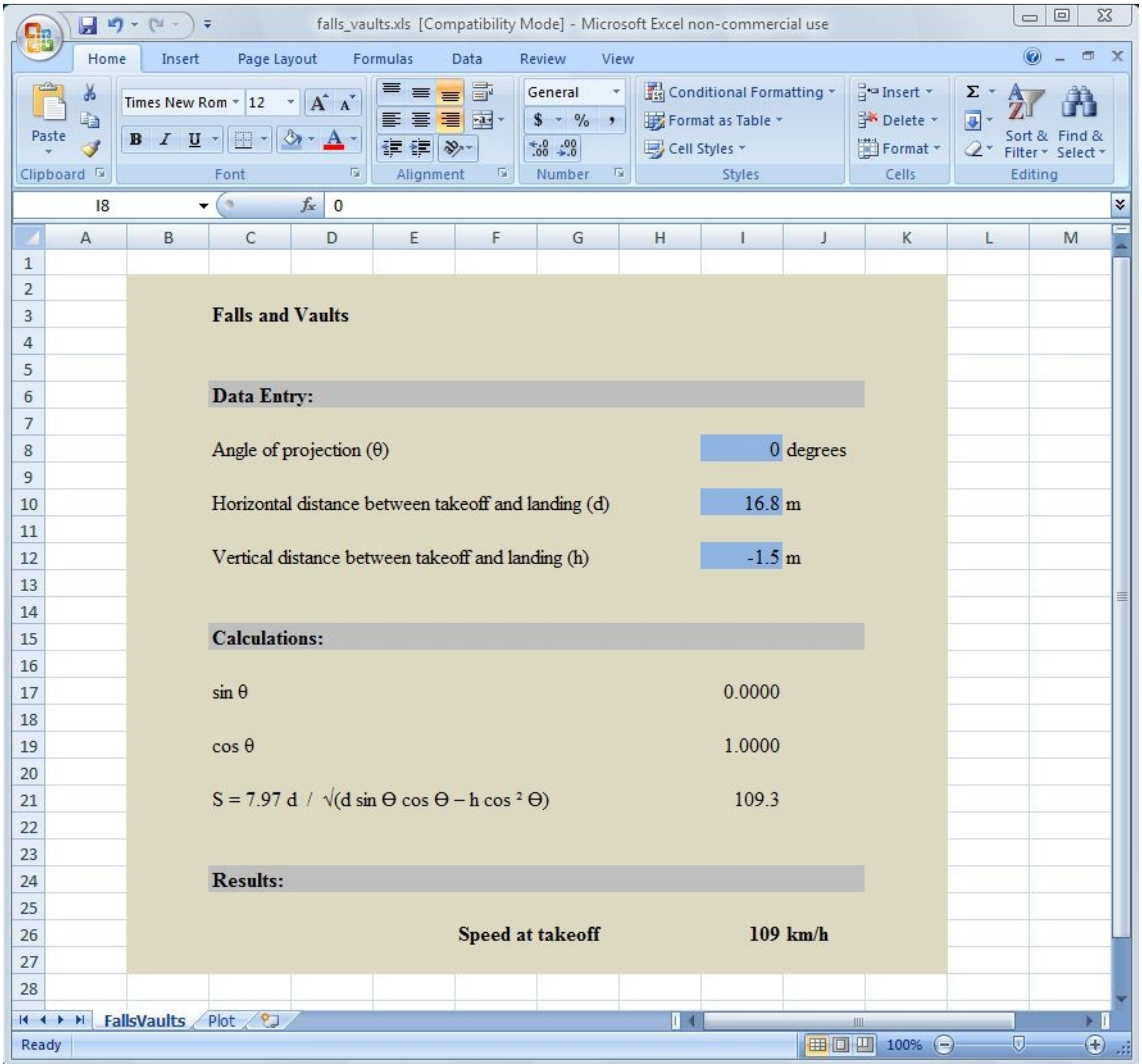

Figure 5. Computer spreadsheet used to calculate takeoff speed

In Excel, an asterisk denotes multiplication so that that the numerator (7.97*I10) is the 7.97 d in Equation 7.

I10*I17*I19 is the value of d sin Θ cos Θ, while I12*I19*I19 is the value of h cos Θ x cos Θ or h $\cos^2 \Theta$.

SQRT, in the denominator, is Excel's square root function, and the nested brackets ensure that the separate terms are calculated and combined in the correct order.

Finally, cell I26 is a copy of the contents of cell I21 (using the Excel expression =I21) with cell I26 being formatted to a number with no decimal places. Thus, our final result is the speed at take-off (V) rounded as a whole number.

It was indicated earlier that Equation 5 can be used to determine the path of a projectile.

$$y = \frac{x \sin \Theta}{\cos \Theta} - \frac{1}{2} g \frac{x^2}{V^2 \cos^2 \Theta} \qquad (5)$$

This equation gives the value of height (y) reached by the projectile at any point (x) down range along the horizontal axis, given a specific speed at take-off (V) and a particular angle of projection (Θ).

Another "calculation" methodology that can be adopted is to use Equation 5 to generate a series of curves for different initial speeds and a given angle of projection. For an object that is projected at the subject angle, the point of landing, corresponding to the measured horizontal (run) and vertical (rise) distances travelled from the origin, can be interpolated, and an approximate initial speed estimated from the series of curves.

This procedure has been reported previously as a graphical solution for investigators who are not sufficiently familiar with the required mathematics to perform the vault calculation. [1]

For example, a series of trajectories for an object projected horizontally (Θ = 0°) over a range of initial speeds, from 20 km/h through 160 km/h, is shown in Figure 6.

Consider our previous example of a vehicle undergoing a fall from a horizontal road onto a roadway allowance, where:

d = 16.8 m
h = -1.5 m
Θ = 0°

In Figure 6, the point d = 16.8 m, h = -1.5 m falls midway between the two curves for initial speeds (V) of 100 km/h and 110 km/h. Careful measurement will show that this point is indeed almost halfway between these two curves. Thus, just by look-up from the chart, we can estimate the vehicle's initial speed at take-off as approximately 110 km/h. (And, if you recall, the calculated speed was 109 km/h.)

Summary

The equations governing falls and vaults, as derived in this paper, are:

Vault formula:

$$S = \frac{7.97\, d}{\sqrt{(d \sin \Theta \cos \Theta - h \cos^2 \Theta)}} \qquad (7)$$

Fall formula:

$$S = \frac{7.97\, d}{\sqrt{(-h)}} \qquad (9)$$

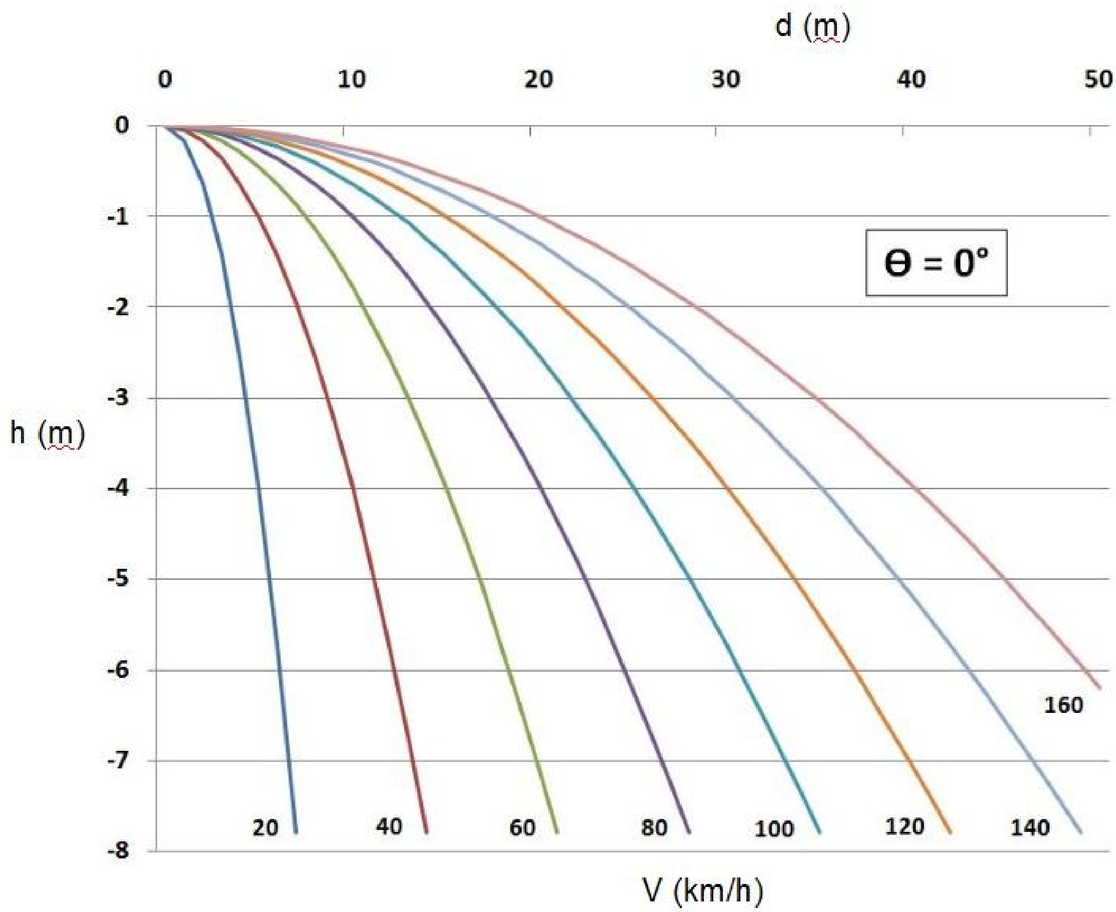

Figure 6. Projectile trajectories for horizontal take-off

Note that both these formulae assume a sign convention where the run (d) is measured positive to the right of the point of take-off, the rise (h) is measured positive above the point of take-off, and the angle of projection (Θ) is positive above the horizontal.

Also, using Equation 7, the initial speed (S) is calculated in km/h since the conversion factor between m/s and km/h is built into the equation's constant.

Apogee formula:

$$h_{max} = \frac{V^2 \sin^2 \Theta}{2g} \qquad (8)$$

In Equation 8, using the initial speed in m/s will give the maximum trajectory height (apogee) in metres.

References

1. German A, Nowak ES, and Green RN; Vehicle Dynamics: Free-Flight Trajectory Analysis; *Proc. 25th Conf. AAAM*; pp. 435-448; October, 1981

Author

Alan German is a research physicist who obtained both a BSc and a PhD from the University of Salford in the United Kingdom. For over 30 years he has been involved in the study of motor vehicle safety through a programme of in-depth investigations of real-world crashes.

Commencing his career as a Research Associate with the Multi-Disciplinary Accident Research Team at The University of Western Ontario, Alan retired as the Chief of the Collision Investigation and Research Division of Transport Canada's Road Safety and Motor Vehicle Regulation Directorate in 2007. He is a Past President and past Executive Director of the Canadian Association of Road Safety Professionals, and a past member of the Board of Directors of the Association for the Advancement of Automotive Medicine.

His research interests include the collision performance of occupant restraint systems, collision investigation and reconstruction techniques, and the application of microcomputers to motor vehicle safety. He has authored or co-authored over eighty publications on a variety of aspects of traffic safety. He was co-author of the paper "The Use of Event Data Recorders in the Analysis of Real-World Crashes" that received the inaugural Dr. Charles H. Miller Award for the best technical paper presented at the 12th. Canadian Multidisciplinary Road Safety Conference.

In 2001 Alan was presented with a U.S. Government Award for Safety Engineering Excellence, and in 2006 was the recipient of a President's Choice Award from the Canadian Association of Technical Accident Investigators and Reconstructionists.